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A SHORT TREATISE

ON THE

CONIC SECTIONS;

IN WHICH

THE THREE CURVES ARE DERIVED FROM A GENERAL DESCRIPTION ON A PLANE,

AND

THE MOST USEFUL PROPERTIES OF EACH ARE DEDUCED FROM A COMMON PRINCIPLE.

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PREFACE.

THE following Treatife was drawn up for the ufe of the author's pupils; and the only motive which induced him to publifh it, was a defire of promoting the ftudy of Conic Sections in this Univerfity, by facilitating the attainment of that ufeful branch of geometry, which has not been fufficiently attended to. Some enter upon Newton's Principia with little or no previous knowledge of conics, taking the common properties of the fections for granted. Others, who have once laboured through fome kind of demonftration, think it fufficient, if they remember the propofitions, although they have forgotten the proofs. Such inattention to a fcience, which is of fo much importance in the prefent fyftem of philofophy, I can attribute only to the want of a concife geometrical treatife.

The authors who have written upon the fub-

ject may be divided into two claſſes; thoſe who have begun with the ſections of the cone; and thoſe who have begun with a deſcription of the curves in plano. Although ſome of the demonſtrations of the latter, who have treated the ſubject geometrically, are ſhort and perſpicuous, yet there are others, upon which depend ſome of the principal properties, that are tedious and difficult. The demonſtrations of the firſt claſs of authors are free from this objection, being in general plain and conciſe; but they have been obliged to introduce ſo many previous propoſitions, concerning the properties of lines touching and cutting conical ſurfaces, in order to arrive at the principal properties of the three ſections, that it requires more time, than can well be ſpared from that portion which is allotted to an academical education, and more reſolution than moſt young men are poſſeſſed of, to go through them. Thoſe of the ſecond claſs, who have treated the ſubject algebraically, have, ſome of them, reduced the whole into a narrower compaſs; but, in their eagerneſs to avoid the charge of prolixity, they have fallen into another more exceptionable fault. The method in which they have deduced ſome of the properties, particularly the relations of the ordinates and abſciſſæ, is extremely operoſe and inelegant; each

ſtep

ſtep in the proofs is ſo little connected with the preceding one, that it is ſcarcely poſſible to retain them in the memory.

I ſhall not detain the reader with any comparison between the two methods, in order to determine which is the beſt; much may be ſaid in favour of both; but the preference ſeems to have been given to the latter, at leaſt in this Univerſity, where mathematical and philoſophical ſtudies are more particularly attended to; yet, with regard to neatneſs and clearneſs of demonſtration, the authors of the latter claſs cannot be compared with ſome of the former. To be convinced of this, let the reader take any one of the principal properties of the ellipſe or hyperbola, and compare the algebraical demonſtration of L' Hoſpital, or Trevigar, the geometrico-algebraical demonſtration of Emerſon, or even the geometrical one of Simſon with that of Hamilton, and he will not heſitate to determine in favour of the latter. But the ſections of the cone, on account of the many interſections of right lines and planes with ſurfaces, and with ſolids, are not eaſily to be comprehended by thoſe who are acquainted only with the elements of geometry.

I had long been perſuaded, that all the known properties of the conic ſections

might be deduced geometrically from a principle which is common to all the ſections, viz. the given ratio between the diſtances of every point in the curve from the focus and the directrix, in the ſame manner as Hamilton has deduced ſome of them in the ſecond book of his treatiſe. I therefore aſſumed that property, which he has demonſtrated, Prop. 11. Book 2. for a definition of a conic ſection, and had made a conſiderable progreſs, before I was fortunate enough to meet with the Elementa Matheſeos of Boſcovich; a work which ſeems to have been little known, or not ſo much eſteemed as it deſerves, although the author is juſtly celebrated for his later productions. In his elements of conic ſections, which have all the advantages of thoſe authors who have begun with the cone, without any of the diſadvantages, I found the plan I had in view in a great meaſure executed. I have, therefore, adopted many of his demonſtrations, with little or no variation; the arrangement of the propoſitions, and ſeveral of the proofs have been much altered; and of ſome I have been obliged to give new demonſtrations, having excluded the harmonical diviſion of right lines, upon which they depended. I have alſo taken from other authors, particularly from the ſecond book of Hamilton, ſuch

ſuch propoſitions as were conformable to the preſent plan. Upon the whole, I have endeavoured to compreſs the ſubject as much as poſſible, and yet not to omit any of the properties, which every one ought to be acquainted with, previous to his entering upon the Principia of Newton, and the branches of natural philoſophy; I have alſo taken care that the demonſtrations ſhould be ſtrictly geometrical, ſuch as the young ſtudent will find no difficulty in underſtanding, provided he be well acquainted with the Elements of Euclid, and plane Trigonometry.

As this treatiſe was deſigned to be an introduction to the Principia, I could not, with propriety, make uſe of the principles contained in the firſt ſection, in comparing the areas of the ſections which have a common axis, or in the quadrature of the parabola; but if the reader be already acquainted with the doctrine of ultimate ratios, he may ſhorten the demonſtration of propoſition 69. in the following manner. It being proved, that the parallelograms in the figure APN are to the parallelograms in the figure AQN in the given ratio of NP to NQ, which is the ſame with that of the conjugate axes, and the parallelograms APN, AQN being ultimately equal to the areas APN, AQN, Lem. 2.

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Newton's Principia, the areas are to each other in the ſame ratio.

The quadrature of the parabola may be demonſtrated from the ſame principles. Let *AN*, Pl. 10. Fig. 88. be the abſciſſa of any diameter of a parabola, and *PNQ* the ordinate; through the point *A* draw *BC* parallel to *PQ*; and through *P*, *Q* draw *PB*, *QC* parallel to *NA*; then the area *PAQ* will be to the parallelogram *PBCQ* as 2 to 3; for let the abſciſſa *AN* be divided into any number of equal parts, of which *ND* is one; through *D* draw *HI* parallel to *PQ*, cutting the parabola in the points *F*, *G*; and through the point *F* draw *KE* parallel to *NA*; take *KR* equal to *KP*, and draw *RL* parallel to *NA*; then the parallelogram *RB* will be double the parallelogram *KB*; and if the number of parts in *AN* be increaſed without limit, the parallelogram *DP* will be equal to the parallelogram *RB*: for the right line *HG* will be ultimately equal to *PQ*, or 2*PN*; and, Prop. 31. the rectangle under *HF*, *HG*, or the rectangle under *PK* and 2*PN*, is to *PH* as the ſquare of *PN* is to *NA*, or *PB*; and alternately, the rectangle under *PK* and 2*PN*, or the rectangle under 2*PK* and *PN*, is to the ſquare of *PN* as *PH* is to *PB*; therefore 2*PK*, or *PR*, is to *PN* as *PH* is to *PB*, and the parallelograms *RB*, *PD* are equiangular;

there-

therefore they are equal, and the parallelogram *PD* is to the parallelogram *KB* as 2 to 1, and the ſum of all the parallelograms in *APN* to the ſum of all in *APB*, or the area *APN* to the area *APB* in the ſame ratio. Therefore the area *APN* is to the parallelogram *ABPN* as 2 to 3, and the area *PAQ* to the parallelogram *PBCQ* in the ſame ratio.

I have only to add that, had it not been for the encouragement, and the liberal aſſiſtance which I have received from the Univerſity, in defraying the expenſe of paper and printing, this little tract would not have appeared in public.

CONIC SECTIONS.

DEFINITIONS.

I. IF any point S be assumed without the line DX, and whilst the line SP revolves about S as a Center, a point P moves in it in such a manner, that its distance from the point S shall always be to PE, its distance from the line DX, in a given ratio, the Curve described by the point P is called a Conic Section; a Parabola, an Ellipse, or an Hyperbola, according as SP is equal to, less, or greater than PE. FIG. 1, 2.

II. The indefinite right line DX is called the Directrix.

III. The point S is called the Focus.

IV. The ratio of SP to PE is called the Determining Ratio.

V. If a line SD be drawn through the Focus perpendicular to the Directrix, which is produced indefinitely, it is called the Axis of the Conic Section.

VI. The point A, where the Curve meets the Axis, is called the Vertex.

VII. A right line LST, drawn through the Focus parallel to the Directrix, and terminated by the Curve in the points L, T, is called the Principal Parameter, or the Latus Rectum.

FIG. 2. COR. 1. SP being greater than PE in the hyperbola, two curves will be described, one on each side of the directrix; which are called opposite hyperbolas.

COR. 2. When the revolving line SP comes into the position SAD, SP, PE will be equal to SA, AD; therefore SA is to AD in the determining ratio.

COR. 3. When the line SP comes into the position SL, or ST, the distance of P from the directrix will be equal to SD, and SL, or ST will be to SD in the determining ratio; and therefore the latus rectum LT is bisected in S.

COR. 4. The latus rectum in the parabola is equal to twice the distance of the focus from the directrix, or to four times its distance from the vertex. For SL is equal to SD, and SA is equal to AD; therefore LT is equal to twice SD, or to four times SA.

PROPOSITION I.

FIG. 3, 4, 5. If two right lines DLQ, DTq be drawn from the point D, where the axis meets the Directrix, through L and T, the extremities of the Latus Rectum, which are produced both ways in the Hyperbola; and through any point P in the Conic Section a line QPq be drawn parallel to the Directrix, meeting the two lines DLQ, DTq in Q and q; the Segment QN, which is intercepted between either of the lines and the axis, will be equal to SP, the distance of P from the Focus.

FOR

FOR the triangles DNQ, DSL are similar, and NQ is to ND as SL is to SD, that is, Cor. 3. Def. in the determining ratio, or as SP to ND; therefore NQ is equal to SP. In the same manner it may be proved that qN is equal to Sp, which is equal to SP, each of them being to ND in the determining ratio.

Cor. 1. If KAG be drawn through the vertex parallel to the directrix, SA will be equal to AK, or AG.

Cor. 2. The lines DLQ, DTq touch the Conic Section in the points L and T. For SNP being a right angled triangle, SP is always greater than PN, except when P is at L, where they coincide; therefore QN is always greater than PN, except when QN coincides with LS; DQ, therefore, meets the curve only in one point L. In the same manner it may be shown that Dq touches the curve in T.

Cor. 3. The angle LDT, contained between the tangents DLQ, DTq, is a right angle in the parabola, an acute angle in the ellipse, and an obtuse angle in the hyperbola. For in the parabola SL is equal to SD, and the angle DSL is a right angle; therefore the angle SDL is half a right angle, and the whole angle LDT a right angle. In the ellipse SL is less than SD, and the angle SDL less than the angle SLD, therefore less than half a right angle, and the whole angle LDT less than a right angle. In the hyperbola SL is greater than SD, and the angle SDL greater than SLD, therefore greater than half a right angle, and the whole angle LDT greater than a right angle.

Fig. 3. Fig. 4. Fig. 5.

PROP. II.

FIG. 3, 4, 5. If from the point *G*, where the right line *KAG*, which is drawn through the Vertex parallel to the Directrix, meets either of the Tangents *DTq*, *DLQ*, a line *GSR* be drawn through the Focus, which is produced both ways in the Hyperbola, it will be parallel to the other Tangent *DLQ* in the Parabola; it will meet it somewhere in *g*, in the direction *GSg*, in the Ellipse, and in the opposite direction in the Hyperbola.

LET *SG* meet the directrix in *X*; and because the triangles *SAG*, *SDX* are similar, and *SA*, Cor. 1. Prop. 1. is equal to *AG*, *SD* will be equal
FIG. 3. to *DX*; but, in the parabola, *SL* is equal to *SD*; it is, therefore, equal and parallel to *DX*; and con-
FIG. 4. sequently, *XGS* is equal and parallel to *DL*. In the ellipse *SL* is less than *SD*, or *DX*; and therefore the lines *DL*, *XS* must meet, when produced,
FIG. 5. in the direction *XGS*. In the hyperbola *SL* is greater than *SD*, or *DX*; and therefore the lines must meet, when produced, in the direction *SGX*.

COR. 1. Because the triangle *SNR* is similar to the triangle *SAG*, *SN* will be equal to *NR*.

FIG. 4, 5. COR. 2. Hence when *Q* coincides with *g*, in the ellipse or opposite hyperbola, *QN* will be equal to *gM*, or *SM*; therefore *SP* will be equal to *SN*; and therefore *SP* will coincide with *SN*, and the curve will meet the axis in the point *M*.

FIG. 4. COR. 3. Hence the whole ellipse is contained

between the lines GK, gk, on one ſide of the directrix.

Cor. 4. In the parabola, NL being always greater than NR, except at the vertex, SP is greater than SN; therefore the curve will meet the axis only in one point A, and it will be extended without limit, on one ſide of the directrix. Fig. 3.

Cor. 5. In the Hyperbola, NL being greater than NR, except at A and M, SP is greater than SN, and the two curves will be extended without limit, on oppoſite ſides of the directrix. Fig. 5.

Cor. 6. The lines KAG, kMg touch the Conic Section in the points A and M. For P and p coincide in theſe points; therefore the right lines KAG, kMg meet the curve only in one point.

Cor. 7. In the ellipſe, SP is the greateſt diſtance from the focus, and SA the leaſt; and thoſe diſtances which are nearer to SM are greater than thoſe which are more remote. For SP is to ND in a given ratio; ND increaſes from A to M, it is the leaſt at A, and the greateſt at M; therefore SP increaſes from A to M in the ſame proportion. Fig. 4.

Cor. 8. In the parabola and hyperbola SA is the leaſt diſtance from the focus, and SP increaſes without limit; and SM is the leaſt diſtance in the oppoſite hyperbola.

DEFINITIONS.

VIII. The tangents DLQ, DTq, which are drawn through the extremities of the Latus Rectum, are called Focal Tangents.

IX. The right line AM, in the Ellipſe and Hyperbola, is called the Tranſverſe Axis, or the Axis Major. Fig. 4, 5.

X. If the Tranſverſe Axis be biſected in C, the point

point C is called the Center of the Ellipſe or Hyperbola.

XI. If a line BCb, which is biſected in C, be drawn perpendicular to the Tranſverſe Axis, and CB, Cb be each of them a mean proportional between SA, SM, the ſegments of the axis intercepted between the focus and the vertices, BCb is called the Conjugale Axis, or the Axis Minor.

XII. A right line PNp, drawn through any point N in the Axis parallel to the Tangent KAG, or perpendicular to the Axis, and terminated by the Curve in the points P and p, is called an Ordinate to the Axis.

XIII. And the Segment of the axis AN, intercepted between the ordinate and the vertex, in all the Sections, as alſo the other ſegment NM in the Ellipſe and Hyperbola, is called an Abſciſſa.

XIV. Any line paſſing through the center of an Ellipſe or Hyperbola, which is terminated both ways by the Curve in the former, and by the oppoſite Curves in the latter, is called a Diameter.

XV. A line drawn through any point in the parabola parallel to the Axis is called a Diameter to the Parabola.

XVI. Any point where a Diameter meets the Curve is called a Vertex to that Diameter.

PROP. III.

The Axis biſects all its Ordinates, and divides the Conic Section into two equal and ſimilar parts.

FIG. 3, 4, 5. LET PNp be any ordinate, meeting the axis in N. Join SP, Sp; and becauſe Sp is equal to

to SP, Prop. 1. and SN is common to the two right angled triangles SNP, SNp, Np will be equal to NP. And because all the ordinates are bisected, if the curve ATp be turned round upon the axis AN, and placed upon ALP, all the points p will coincide with all the points P, and the curve ATp with the curve ALP.

PROP. IV. PROB.

The Focus, Directrix, and the Determining Ratio being given, to describe the Conic Section.

LET DX be the directrix, and S the focus. Through the point S draw SD perpendicular to DX, which produce indefinitely. Draw LST parallel to DX; and take SL and ST to SD in the determining ratio. Then LST, Cor. 3. Def. is the latus rectum. Join DL, DT, which must also be produced indefinitely. Take DX, in the directrix, equal to DS, and join XS, cutting the line DT in G, which, Prop. 2. will be parallel to the line DL in the parabola; it will meet it in some point g, in the direction DL, in the ellipse, and in the opposite direction in the hyperbola. Through the points G and g draw KAG, gMk parallel to the directrix, meeting the two lines DLg, DTk, and the axis in the points K, G, A, and g, k, M; the points A and M will, therefore, be the vertices of the axis. Through any point N, in the axis, between A and M in the ellipse, any where on the same side of A with S in the parabola, and any where except between A and M in the hyperbola, draw the line QNq parallel to the directrix; and from

FIG. 3, 4, 5.

from the center S, with a radius equal to QN, describe a circle, cutting the line Qq in the points P, p; and join SP, Sp, which are each of them equal to QN; and therefore the points P and p will be in the curve, by the first proposition.

COR. I. If the latus rectum, and the distance of the focus from the vertex be given, the Conic Section may be described. For let S be the focus; and through S draw the indefinite right line ASN, in which take SA equal to the given distance from the vertex. Draw LST perpendicular to ASN; and take SL, ST each of them equal to half the latus rectum. Through A draw KAG parallel to LT; and take AG equal to AS. Join TG, which produce to meet SA in D: Join DL and GS, which produced, will meet in the point g in the direction DL, in the opposite direction, or they will be parallel; in the first case the Section will be an ellpise, in the second an hyperbola, and in the third a parabola, and the curve may be described as before.

COR. 2. Hence if two right lines DQ, Dq be inclined to each other at any given angle, which is bisected by the line DN, and another line Qq, which is perpendicular to DN, move parallel to itself, and intersect the lines SP, Sp, revolving about any point S in the line DN, in such a manner, that SP, Sp shall always be equal to QN or qN, the points of intersection P, p will describe a Conic Section, which, Cor. 3. Prop. I. will be a parabola, an ellipse, or an hyperbola, according as the angle QDq is equal to, less, or greater than a right angle.

PROP.

PROP. V.

The square of the semi-ordinate to the axis, in the Parabola, is equal to the rectangle under the latus rectum and the abscissa.

BECAUSE the line Qq is bisected in N, the rectangle QRq, together with the square of RN, is equal to the square of QN, which is equal to the square of SP, Prop. 1. or to the squares of SN, PN, or of RN, PN, Cor. 1. Prop. 2. and if from each of these be taken the square of RN, the remaining rectangle QRq will be equal to the square of PN. But QR is equal to SL, and Rq is equal to RN and Nq, or to SN, Nq; and the angle NDq being half a right angle, Cor 3. Prop. 1. NqD is also half a right angle, and Nq is equal to ND; therefore Rq is equal to SN and ND, that is to SD and twice SN, or to twice SA and twice SN, Cor. 2. Def. which is equal to twice AN, therefore the rectangle QRq is equal to the rectangle under SL and twice AN, or to the rectangle under AN and twice SL, or LT; and therefore the square of PN is equal to the rectangle under AN, LT. FIG. 3.

COR. 1. The latus rectum being constant, the abscissa will vary as the square of the ordinate.

COR. 2. The parabola recedes from the axis without limit. For the abscissa increases without limit, and therefore the square of the semi-ordinate, which varies as the abscissa, will also increase without limit.

COR. 3. Any line, which is drawn parallel to the axis of the parabola, will cut the curve only in one

point. For if it be ſuppoſed to cut the curve in more points than one, the ſemi-ordinates drawn through the points of interſection would be equal, when the abſciſſæ are unequal, which is impoſſible.

LEMMA I.

If four ſtraight lines be proportionals, and any other four proportionals, the rectangle under the firſt and fifth will be to the rectangle under the ſecond and ſixth as the rectangle under the third and ſeventh to the rectangle under the fourth and eighth.

Fig. 6. LET *AB* be to *CD* as *EF* to *GH*, and *BI* to *DK* as *FL* to *HM*; and let *AI* be the rectangle under *AB*, *BI*; *CK* the rectangle under *CD*, *DK*; *EL* the rectangle under *EF*, *FL*; and *GM* the rectangle under *GH*, *HM*; then *AI* will be to *CK* as *EL* to *GM*. For in *DK*, *HM*, produced if neceſſary, take *DN*, *HO* ſuch, that *AB* ſhall be to *CD* as *DN* to *BI*, and *EF* to *GH* as *HO* to *FL*; and complete the rectangles *CN*, *GO*. Then the rectangle *CN* is equal to the rectangle *AI*, and the rectangle *GO* equal to the rectangle *EL*. But, *AB* being to *CD* as *EF* to *GH*, and as *DN* to *BI*, *DN* is to *BI* as *EF* to *GH*, or as *HO* to *FL*. But *BI* is to *DK* as *FL* to *HM*; therefore *DN* is to *DK* as *HO* to *HM*; and the rectangle *CN* being to the rectangle *CK* as *DN* to *DK*, and the rectangle *GO* to the rectangle *GM* as *HO* to *HM*, *CN* is to *CK* as *GO* to *GM*; therefore the rectangle *AI* is to the rectangle *CK* as the rectangle *EL* to the rectangle *GM*.

Cor.

COR. If AB be to CD as EF to GH, the ſquare upon AB will be to the ſquare upon CD as the ſquare upon EF to the ſquare upon GH; for, if any of the correſponding terms, as AB, BI, be equal to each other, the rectangle AI becomes the ſquare upon AB.

PROP. VI.

The ſquare of the ſemi-ordinate to the axis, in the ellipſe and hyperbola, is to the rectangle under the abſciſſæ as the ſquare of the conjugate axis is to the ſquare of the tranſverſe axis.

THROUGH the point G draw GVW in the ellipſe, and $VGWV$ in the two hyperbolas parallel to the tranſverſe axis AM. Then, becauſe the lines KAG, QNq, gMk are parallel, FIG. 4, 5

$QR : KG :: gR : gG :: VW : GW :: NM : AM$, &

$Rq : gk :: GR : Gg :: GV : GW :: AN : AM$;

therefore, Lem. 1. the rectangle QRq is to the rectangle KG, gk as the rectangle ANM to the ſquare of AM. But it may be proved, in the ſame manner as in the laſt propoſition, that the rectangle QRq is equal to the ſquare of PN; and GK being equal to twice SA, and gk equal to twice SM, Cor. 1. Prop. 2. the rectangle KG, gk is equal to four times the rectangle ASM, or to four times the ſquare of BC, Def. 11. which is equal to the ſquare of Bb; therefore the ſquare of PN is to the ſquare of Bb as the rectangle ANM to the ſquare of AM; and alternately, the ſquare of PN is to the rectangle ANM as the ſquare of Bb to the ſquare of AM.

Cor. 1. Because AM, Bb are bisected in C, the square of PN is to the rectangle ANM as the square of BC to the square of AC.

Cor. 2. The axes being constant, the square of the semi-ordinate varies as the rectangle under the abscissæ.

Cor. 3. The conjugate axis in the ellipse is terminated by the curve: for, when the ordinate passes through the center, the rectangle under the abscissæ is equal to the square of half the transverse axis; and therefore the square of the semi-ordinate is equal to the square of half the conjugate axis, and the ordinate is equal to the conjugate axis.

Cor. 4. The two hyperbolas recede from the axis without limit: for the two abscissæ increase without limit; and therefore the square of the semi-ordinate will increase without limit.

Cor. 5. Those ordinates which are at equal distances from the center of the ellipse and the two hyperbolas are equal; and those which are nearer to the center are greater in the ellipse, and less in the hyperbola, than those which are more remote: for the abscissæ which are at equal distances from the center are equal; and the rectangle ANM being equal to the difference of the squares of CM, CN, it will increase in the ellipse, and decrease in the hyperbola, as CN decreases, that is, the nearer PN is to the center.

Cor. 6. Any line, which is drawn parallel to the axis of the hyperbola, will cut each of the opposite curves only in one point; for, if it be supposed to cut either of the curves in more points than one, the ordinates which are drawn through the points of intersection would be equal, when the distances from the center are unequal, which is impossible.

PROP.

PROP. VII.

The latus rectum of the ellipſe and hyperbola is a third proportional to the tranſverſe and conjugate axes.

FOR the ſquare of AC is to the ſquare of BC, Cor. 1. Prop. 6. as the rectangle ASM, or the ſquare of BC, is to the ſquare of SL; therefore AC is to BC as BC to SL, and $2AC$ to $2BC$ as $2BC$ to $2SL$, that is, AM to Bb as Bb to LT. Fig. 4, 5.

Cor. Hence the rectangle under the abſciſſæ is to the ſquare of the ſemi-ordinate as the tranſverſe axis to the latus rectum: for AM is to LT as the ſquare of AM to the ſquare of Bb, that is, as the rectangle ANM to the ſquare of PN.

PROP. VIII.

The ſquare of half the conjugate axis, in the ellipſe and hyperbola, is equal to the difference of the ſquares of half the tranſverſe axis and the diſtance of the focus from the center.

BECAUSE AM is biſected in C, the rectangle ASM is equal to the difference of the ſquares of AC and SC; but the rectangle ASM, Def. 11. is equal to the ſquare of BC; therefore the ſquare of BC is equal to the difference of the ſquares of AC and SC. Fig. 7, 8.

Cor. 1. If a line SB be drawn from the focus of the ellipſe to the vertex of the conjugate axis, it Fig. 7.
will

will be equal to half the tranſverſe axis: for, the ſquare of BC being equal to the difference of the ſquares of AC, SC, the ſquare of AC will be equal to the ſum of the ſquares of BC, SC, or to the ſquare of SB; therefore SB is equal to AC.

Fig. 8. Cor. 2. The diſtance between the vertices of the two axes, in the hyperbola, is equal to the diſtance of the focus from the center: for, the ſquare of BC being equal to the difference of the ſquares of SC and AC, the ſquare of SC will be equal to the ſum of the ſquares of AC, BC, or to the ſquare of AB; therefore AB is equal to SC.

PROP. IX.

The conjugate axis biſects all lines drawn parallel to the tranſverſe axis, which are terminated by the ellipſe, and by the oppoſite hyperbolas. Thoſe lines which are equally diſtant from the center are equal; and thoſe which are nearer to the center are greater in the ellipſe, and leſs in the hyperbola, than thoſe which are more remote.

Fig. 9, 10. Take CN any diſtance from the center, between C and A in the ellipſe, and in CA produced in the hyperbola. Take CR, on the other ſide of the center, equal to CN; and through the points N, R draw the ordinates PNp, QRq; join PQ, pq, and let them meet the conjugate axis in n and r. Then, becauſe the ordinates Pp, Qq are equal, Cor. 5. Prop. 6. and they are biſected in N and R, the lines PQ, NR, pq are equal and parallel; and becauſe Pn, pr are equal to NC, and Qn,

qr equal to RC, or NC, PQ, pq are bisected in n and r; and they are at equal distances from the center, because Cn, Cr are equal to PN, Np. Lastly, as Cn decreases PN decreases, and therefore CN, Cor. 5. Prop. 6. increases in the ellipse, and decreases in the hyperbola; but Pn is equal to CN, Pn therefore increases in the former, and decreases in the latter, as Cn, its distance from C, decreases.

Cor. 1. The conjugate axis divides the ellipse into two equal and similar parts: the two opposite hyperbolas are equal and similar: and the ellipse and hyperbola have each of them another focus and directrix, which have the same properties as the former. Take CH, on the other side of the center, equal to CS, and Cd equal to CD: through d draw xde perpendicular to cd, and let it meet the lines PQ, pq in e, x, and join HQ. Then if the whole figure $nQMqr$ be turned round upon the axis Bb, and placed upon $nPApr$, nQ, rq will coincide with nP, rp, and all the points Q, q in the curve QMq with all the points P, p in PAp. The straight line xde will also coincide with XDE, the point H with S, and the lines HQ, Qe with the lines SP, PE; therefore HQ is always to Qe in the same ratio that SP is to PE.

Cor. 2. Suppose the line EPQ, Fig. 9. which is always parallel to DAM, to move from the center towards B: then, when Cn becomes equal to CB, the points P, Q will coincide in the point B; and the line EPQ, which meets the curve only in one point, will be a tangent to the ellipse; and therefore the ordinates to the conjugate axis are parallel to the tangent at its vertex, Fig. 9.

PROP.

PROP. X.

The ſquare of the ſemi-ordinate to the conjugate axis, in the Ellipſe, is to the rectangle under the abſciſſæ as the ſquare of the tranſverſe axis is to the ſquare of the conjugate axis.

Fig. 9. FOR PnQ being parallel to AM, it is perpendicular to BCb, and it is biſected in n, by the laſt Prop. it is therefore an ordinate to Bb: and the ſquare of PN, or Cn, is to the ſquare of BC as the rectangle ANM to the ſquare of AC, Prop. 6. therefore the difference of the ſquares of BC, Cn is to the ſquare of BC as the difference of the ſquare of AC and the rectangle ANM is to the ſquare of AC; but the difference of the ſquares of BC, Cn is equal to the rectangle Bnb, and the difference of the ſquare of AC and the rectangle ANM is equal to the ſquare of CN or Pn; therefore the rectangle Bnb is to the ſquare of CB as the ſquare of Pn to the ſquare of AC; and alternately, and by inverſion, we have the ſquare of Pn to the rectangle Bnb as the ſquare of AC to the ſquare of BC, or as the ſquare AM to the ſquare of Bb.

PROP. XI.

The tranſverſe axis, in the Ellipſe and Hyperbola, is to the diſtance between the directrices in the determining ratio.

FOR

FOR SA is to AD as SM to MD, and alternately, SA is to SM as AD to MD; therefore by compofition in the ellipfe, and by divifion in the hyperbola, AM is to SA as Dd to AD, and alternately, AM is to Dd as SA to AD; that is, Cor. 2. Def. in the determining ratio. FIG. 9, 10.

COR. 1. Hence, Dd and AM being bifected in G, AC is to CD in the determining ratio.

COR. 2. The diftance between the foci is to the tranfverfe axis in the determining ratio. For SM is to MD as SA to AD, or as HM to AD; and alternately, SM is to HM as MD to AD; and by divifion in the ellipfe, and by compofition in the hyperbola, SH is to HM as AM to AD; and alternately, SH to AM as HM to AD, or as SA to AD.

COR. 3. Hence the diftance between the foci, the tranfverfe axis, and the diftance between the directrices are continual proportionals.

COR. 4. Hence, if the tranfverfe axis and the foci of an ellipfe or hyperbola be given, the determining ratio is given.

PROP. XII.

All the diameters of an Ellipfe or Hyperbola are bifected in the center.

FROM any point P in the curve draw PC to the center, and PN perpendicular to the axis. Take Cn, on the other fide of the center, equal to CN; and through the point n draw nG parallel to NP, but on the oppofite fide of the axis, which produce till it meet the curve in G, and join CG. Then, becaufe Cn is equal to CN, the femi-ordinates Gn, PN will be equal, Cor. 5. Prop. 6. and FIG. 7, 8.

 the

the angles at N and n being each of them a right angle, the triangle CnG will be equal to the triangle CNP; therefore CG is equal to CP, and the angle nCG equal to the angle NCP; and therefore PCG is one ſtraight line, which is biſected in C.

DEFINITION XVII.

Two conic ſections, or two ſegments of conic ſections are ſaid to be ſimilar, when a rectilinear figure may be inſcribed in one of them, ſimilar to any rectilinear figure which is inſcribed in the other.

PROP. XIII.

If the angles contained between the focal tangents be equal, or the determining ratio be the ſame in two conic ſections, the ſections will be ſimilar.

Fig. 3, 4, 5. LET there be two parabolas, two ellipſes, or two hyperbolas, in which the angles contained between the focal tangents are equal; then SDL, which is half of the angle TDL, will be the ſame in both curves, and the ratio of SL to SD, or the determining ratio will be the ſame. And if SP makes the ſame angle with the axis in both, the ratio of SP to SN will be the ſame; but the ratio of SP to ND is the ſame; therefore SP is to SD, the ſum or difference of ND, SD, in the ſame ratio in the two ſections; and alternately, SP in one is to SP in the other as SD in the former to SD in the latter, that is in a given ratio. Fig. 5. Therefore let the two ſections PAQ, paq be placed in ſuch a man-

ner that the foci may coincide in *S*, and that *SA* may be the axis of both: Let *PBACQ* be any rectilinear figure inscribed in one of them; join *SP*, *SB*, *SA*, &c. and let them meet the other section in the points *p*, *b*, *a*, *c*, *q*; and join *pb*, *ba*, *ac*, &c. also join *pq*. Then the figure *pbacq* will be similar to *PBACQ*. For *SP* is to *Sp* as *SB* to *Sb*; and alternately, *SP* is to *SB* as *Sp* to *Sb*, and the angle *PSB* is common to the two triangles *PSB*, *pSb*; therefore the triangles are similar. In the same manner it may be proved that all the other triangles are similar, and consequently the whole rectilinear figure *pbacq* similar to the rectilinear figure *PBACQ*, and therefore the section *paq* is similar to the section *PAQ*.

COR. 1. The determining ratio in the parabola being a ratio of equality, all parabolas are similar.

COR. 2. Two ellipses or hyperbolas are similar when the axes have the same ratio to each other in both. For *SC* being to *CA* in the determining ratio, which is the same in both, the square of *AC* is to the square of *SC*, or to the difference of the squares of *AC*, *SC*, or to the square of *BC*, Prop. 8. in the same ratio; and consequently *AC* is to *BC*, or *AM* to *Bb* in the same ratio in both the sections. FIG. 4, 5.

PROP. XIV.

If from any point in the Ellipse or Hyperbola two right lines be drawn to the foci, the sum of these lines in the Ellipse, and their difference in the Hyperbola is equal to the transverse axis.

PL. IV. FIG. 27, 28. LET P be any point in the ellipse or hyperbola; and let S and H be the two foci. Join PS, PH; and through the point P draw the line EPe, Fig. 27. and PEe, Fig. 28. parallel to the axis; and let it meet the two directrices in E and e. Then PE, Pe will be perpendicular to the directrices, and SP will be to PE in the determining ratio, or as HP to Pe; and alternately, SP is to HP as PE to Pe; therefore the sum of SP, PH, Fig. 27. and their difference, Fig. 28. is to SP as Ee, or Dd, to PE; and alternately, the sum or difference of SP, PH is to Dd as SP to PE, or, by the eleventh proposition, as AM to Dd; therefore the sum of the lines SP, PH in the ellipse, and their difference in the hyperbola is equal to AM, the transverse axis.

PROP. XV. PROB. II.

Two right lines being given, which bisect each other at right angles; to describe an Ellipse, or an Hyperbola, of which the given lines shall be the axes.

FIG. 11. FIRST, let the curve be an ellipse, in which case the given right lines must be unequal. Let AM, Bb be the given lines, and let Bb be less than AM. From the center B, with a radius equal to AC, describe a circle, cutting the line AM in the points S and H. Take a string equal in length to AM, and fix the extremities of it in the points S and H; and, by means of a pin at P, let the string be stretched, and let the pin be carried round till it return to the same point. The point P will describe an ellipse, of which AM, Bb are the axes.

For

FIG. IV.

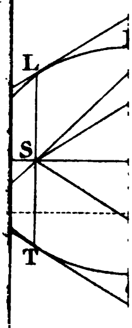


FIG. II.

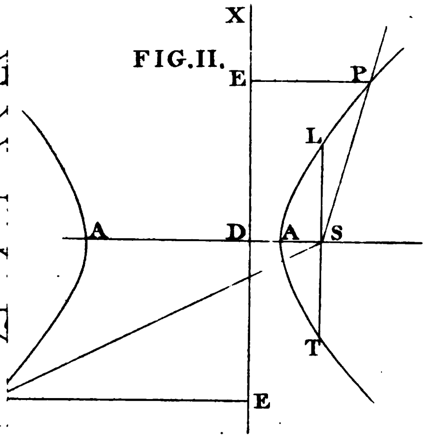
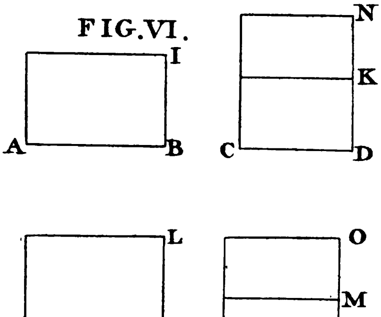


FIG. VI.



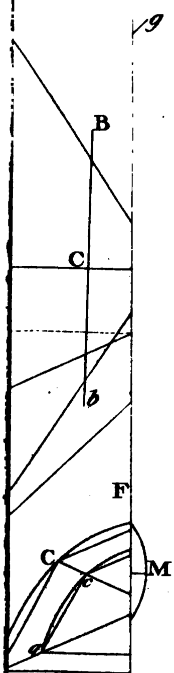
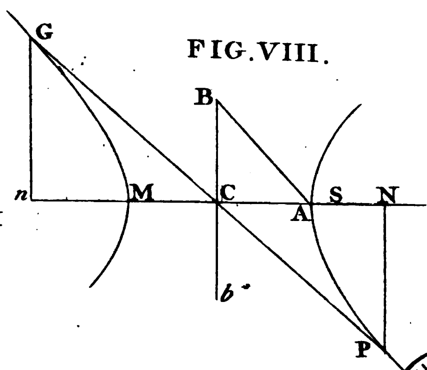


FIG. VIII.



For ſuppoſe the ellipſe deſcribed, of which S is the focus, A the neareſt vertex, and the ratio of SH to AM the determining ratio, which may be done, by the fourth propoſition; then AM will be the tranſverſe axis, Cor. 2. Prop. 11. and Bb the conjugate axis of that ellipſe, Prop. 8. and the point P will be in the ellipſe in every part of its revolution. For if not, let HP meet the curve in ſome other point G, nearer to H, or further from it; then the ſum of the lines SG, GH would be equal to AM, or to SP, PH, which is impoſſible.

Secondly, let AM, Bb be any two given right lines, biſecting each other at right angles in C; and let the curve be an hyperbola. Join AB; and from the center C take CS and CH, in AM produced both ways, equal to AB. At the point H let the end of a ruler be fixed, ſo that it may move freely round this point as a center; and let a ſtring be taken, the length of which the ruler exceeds by a line equal to AM; let one end of the ſtring be fixed at L, and the other in the point S; apply the ſtring, by means of a pin at P, to the ſide of the ruler LH; and let the ruler be moved about the center H, whilſt the ſtring is conſtantly applied, and kept cloſe to the ruler by the pin at P. Then, the difference between the whole length of the ſtring SPL and the ruler HL being equal to AM, the difference between HP and PS will be equal to AM; and the point P will deſcribe one of the oppoſite hyperbolas, of which AM, Bb are the axes. For ſuppoſe the hyperbola deſcribed of which S is the focus, A the vertex, and the ratio of SH to AM the determining ratio; then AM will be the tranſverſe axis, Cor. 2. Prop. 11. and Bb the conjugate axis of that hyperbola, Prop .8. and the point P will be in the hyperbola in every part of its revolution. FIG. 12.

For

For if not, let the line SP meet the curve in ſome other point G, nearer to S, or further from it. In the firſt caſe the difference between HG, GS is equal to AM; therefore HG is equal to AM and SG; and HG, GP is equal to AM and PS, which is equal to HP; therefore HP is equal to HG, GP, which is impoſſible. In the ſecond caſe HP is equal to AM and SP; and HP, PG equal to AM and SG, or to HG, which is alſo impoſſible. Therefore the point P muſt be in the curve.

PROP. XVI. PROB. III.

Two right lines being given, one of which is biſected by the other at right angles; to deſcribe a Parabola, in which the right line biſected ſhall be an ordinate, and the other line the axis.

FIG. 13. LET AC, Bb be the two given right lines, one of which Bb, which is perpendicular to AC, is biſected in C. Find a third proportional to AC, CB; and produce CA to D, ſo that AD may be a fourth part of that third proportional; and take AS equal to AD. Through D draw DEX perpendicular to DAC; and let a ruler, the ſides of which HE, EL are perpendicular to each other, be placed in the plane CDX, ſo that the ſide EL may be applied to DX; and take a ſtring equal in length to the ſide HE, one extremity of which muſt be fixed at H, and the other at S; and let part of the ſtring be applied, by means of a pin P, to the ſide of the ruler HE; and whilſt the ſide EL moves along DX, let the ſtring be ſtretched by the pin, and con-

conſtantly applied to *HE*. Then, becauſe the whole length of the ſtring *HPS* is equal to *HE*, the part *SP* will be always equal to *PE*; therefore the point *P* will deſcribe a parabola, by the firſt definition, of which *AC* is the axis, *S* the focus, and *DX* the directrix; and *BCb* will be an ordinate of that parabola, becauſe it is perpendicular to the axis, and *CB* is a mean proportional between the abſciſſa *AC* and four times *AS*, or the latus rectum.

PROP. XVII.

If any point be taken within a conic ſection, its diſtance from the focus will be to its diſtance from the directrix in a leſs ratio than the determining ratio; and if the point be taken without the ſection, its diſtance from the focus will be to its diſtance from the directrix in a greater ratio.

Part 1. LET the point *L* be taken any where within the ſection which is on the ſame ſide of the directrix with the focus, Fig. 14. or within the oppoſite ſection, Fig. 15. Join *LS*, and let it meet the curve in *P*. Draw *LM*, *PE* perpendicular to the directrix. Join *SE*, and let it meet the line *LM* in *F*. Then, becauſe the triangles *SLF*, *SPE* are ſimilar, *SL* is to *LF* as *SP* to *PE*; but *SL* is to *LM* in a leſs ratio than *SL* to *LF*; and therefore in a leſs ratio than the determining ratio. FIG. 14, 15.

Part 2. Let the point *L* be taken without the conic ſection, but on the ſame ſide of the directrix with FIG. 14.

with *S*. Join *LS*, which will cut the curve in *P*; and draw *LM*, *PE* perpendicular to the directrix. Join *SM*, cutting the line *PE* in *G*; and because the triangles *SLM*, *SPG* are similar, *SL* is to *LM* as *SP* to *PG*, which is a greater ratio than that of *SP* to *PE*. Secondly, let the point *L* be on the other side of the directrix, and let the section be an ellipse or a parabola. Join *LS*, cutting the directrix in *N*, and the cvrve in *P*. Draw *LM*, *PE* perpendicular to the directrix; and join *SE*, which produce till it meet *ML* produced in *O*. Then *SL* is to *LO* as *SP* to *PE*; therefore *LO* is not less than *SL*; and *SL* is greater than *LM*; therefore *LO* must be greater than *LM*; and *SL* is to *LM* in a greater ratio than *SL* to *LO*, or *SP* to *PE*.
Fig. 15. Lastly, let the point *L* be taken any where between the directrix and the opposite hyperbola. Draw *LE* perpendicular to the directrix, which produce till it meet the opposite hyperbola in *P*; and join *SP*. Through *P* draw *PF* parallel to the directrix, meeting *SL* produced in *F*; and draw *FM* parallel to *PE*. Join *SE*, and produce it till it meet *FM* in *G*. Then it is evident that *SF* cannot be less than *SP*; and *SL* is to *LE* as *SF* to *FG*, that is, in a greater ratio than *SF* to *FM*, or than *SP* to *PE*.

Cor. Hence, if the distance of any point from the focus of a conic section be to its distance from the directrix in the determining ratio, that point is in the section; and if its distance from the focus be to its distance from the directrix in a greater, or in a less ratio, the point will be without, or within the conic section.

PROP.

PROP. XVIII. PROB. IV.

The focus, the directrix, and the determining ratio being given; to find the points in which a straight line passing through the focus, which is given in position, meets the conic section, or opposite sections.

IF the line which passes through the focus be parallel to the directrix, it will coincide with the latus rectum, and the proposition is manifest. If the line *SQ* be not parallel to the directrix, let it meet it in some point *Q*. Take *QH* and *QG* in the directrix, on opposite sides of *Q*, each of them equal to *QS*. Draw the latus rectum *LST*; and join *LG*, *LH*; and produce *LH* till it meet *QS* produced, if possible, in *p*. The points *P*, *p*, in which *LG* and *LH* meet the line *QS*, will be in the conic section. For draw *PE*, *pe* perpendicular to the directrix; and because the triangles *SPL*, *PQG* are similar, as also the triangles *PQE*, *SQD*, *SP* is to *SL* as *PQ* to *QG*, or *QS*, or as *PE* to *SD*; and alternately, *SP* is to *PE* as *SL* to *SD*, that is, in the determining ratio; and therefore, Cor. Prop. 17. the point *P* is in the curve. And because the triangles *pSL*, *pQH* are similar, as also the triangles *pQe*, *SQD*, *Sp* is to *SL* as *Qp* to *QH*, or *QS*, or as *pe* to *SD*; and alternately, *Sp* is to *pe* as *SL* to *SD*; therefore *p* is also in the curve; and the line *QS* meets the curve in the points *P* and *p*. FIG. 16, 17.

COR. 1. Every line which passes through the focus of the parabola will meet the curve in two points, except that which is perpendicular to the directrix. For it is evident that it will meet the

FIG. 16. curve in some point P, between the latus rectum and the directrix, in all the conic sections; and when Q coincides with D, SQ will be equal to SL; therefore SL is equal to QH, and the lines QS, HL are parallel; but if Q does not coincide with D, SQ will be greater than SD; and therefore QH greater than SL, and the lines QS, HL will meet in some point p, which is in the curve, by the proposition.

COR. 2. Every line which passes through the focus of an ellipse will meet the curve in two points.

COR. 3. If a line passing through the focus of an hyperbola be inclined to the directrix at such an angle, that the radius is to the sine of it in the determinig ratio, it will meet the curve only in one point: if it be inclined at a less angle, it will meet the same hyperbola in two points; and if it be inclined at a greater angle, it will meet each of the opposite curves in one point.

FIG. 16, 17. First, let the radius be to the sine of the angle SQD in the determining ratio; then SQ will be to SD as SL to SD; therefore SQ, or QH, will be equal to SL, and the lines SQ, LH will be parallel.

FIG. 16. Secondly, let the line SQ be inclined to the directrix at a less angle; then SQ is to SD in a greater ratio than SL to SD; therefore SQ, or QH, is greater

FIG. 17. than SL; and QS, HL will meet in some point p in the direction QS. Lastly, let the angle SQD be greater; then SQ is to SD in a less ratio than SL to SD; therefore SQ, or QH, is less than SL; and the lines SQ, LH will meet in the opposite curve.

Pl. 2.
Page 26.

G. IX.

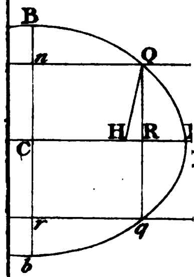


FIG. X.

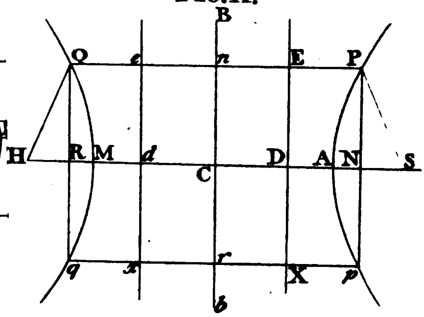


FIG. XVI.

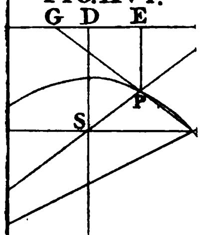


FIG. XIII.

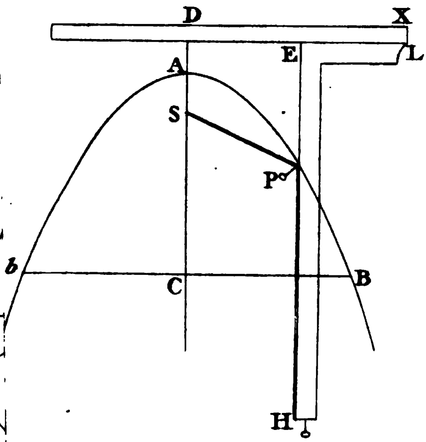
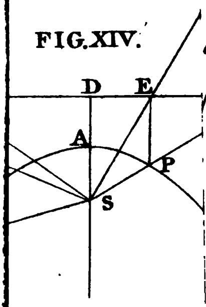


FIG. XIV.



G. XV.

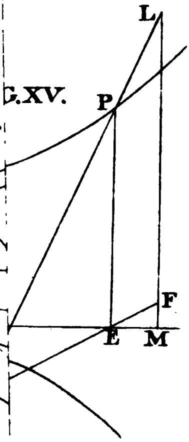
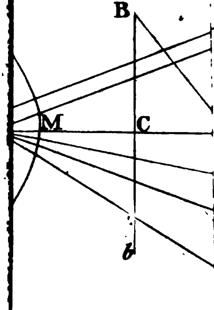


FIG. XII.



DEFINITION XVIII.

If from the center *C*, at the diſtance *CA*, half the transverſe axis, a circle be deſcribed, cutting the directrix of the hyperbola in the points *H*, *h*, and lines be drawn from the center through the points of interſection, theſe lines are called the Aſymptotes. FIG. 18.

PROP. XIX.

If lines be drawn from the focus to the points in which the aſymptotes cut the directrix, they will be perpendicular to the aſymptotes: and the angle contained between the aſymptote and the directrix is ſuch, that the radius is to the ſine of it in the determining ratio.

LET the aſymptotes *Ca*, *CG* cut the directrix in the points *H*, *h*; and join *SH*, *Sh*. Then *CS* is to *CA* as *CA* to *CD*, Cor. 3. Prop. 11. but *CH* is equal to *CA*, therefore *CS* is to *CH* as *CH* to *CD*, and the angle *SCH* is common to the two triangles *SHC*, *CDH*; therefore the triangles are ſimilar, and the angle *SHC* is equal to the angle *CDH*, which is a right angle. In the ſame manner it may be proved that the angle *ShC* is a right angle. Secondly, *CA*, or *CH*, is to *CD* in the determining ratio; and, *CH* being made radius, *CD* is the ſine of the angle *CHD*. FIG. 18.

COR. 1. Hence, if a line be drawn through the focus parallel to an aſymptote, it will cut the curve only in one point.

FIG. 19. COR. 2. If a line PG be drawn from any point P in the hyperbola, or in the opposite curve, parallel to the asymptote, meeting the directrix in G, PG will be equal to PS. From P draw PE perpendicular to the directrix; and because the angle PGE is equal to the angle CHD, PG is to PE in the determining ratio, or as SP to PE; therefore PG is equal to SP.

COR. 3. Hence, if the focus, the directrix, and the position of the asymptote be given, the hyperbola may be described, by means of a ruler and string, in the same manner as the parabola is described, Prop. 16. provided the sides of the ruler be inclined to each other in the same angle as the asymptote and directrix.

PROP. XX.

The asymptotes never meet the curve: but any other line drawn parallel to an asymptote will meet one of the hyperbolas.

FIG. 19. FOR if it be possible, let the asymptote meet the curve in the point R. Join RS, and draw RN perpendicular to the directrix. Then, by the preceding proposition, HR is to RN in the determining ratio, or as RS to RN; therefore RS is equal to RH, and the angle RSH is equal to the angle RHS; which is impossible, the angle RHS being a right angle. In the same manner it may be proved that it cannot meet the opposite curve.

Let any other line GP be drawn parallel to the asymptote; and first let it be nearer to the focus. Join SG, and produce it till it meet the asymptote CH in I; then the angle SGP is equal to the angle SIH,

SIH, which is leſs than the angle *SHR*, a right angle; therefore, the angle *SGP* being leſs than a right angle, if the angle *GSP* be made equal to *SGP*, *SP*, *GP* will meet, when produced, ſomewhere in *P*, which is a point in the curve. For draw *PE* perpendicular to the directrix; and the angle *PGE* being equal to the angle *CHD*, *PG* is to *PE* in the determining ratio; therefore *SP* is to *PE* in the ſame ratio, and *P*, Cor. Prop. 17. is in the hyperbola. Secondly, let *gp* be drawn parallel to the aſymptote, at a greater diſtance from the focus. Join *Sg*, cutting the aſymptote in *i*; then the angle *Sgp* is equal to the angle *SiH*, which is leſs than the angle *SHi*, a right angle: if therefore the angle *gSp* be made equal to *Sgp*, the lines *Sp*, *gp* will meet, when produced, in ſome point *p*, which is in the oppoſite hyperbola; for the angle *pge* being equal to *CHD*, *pg* is to *pe* in the determining ratio; and therefore *Sp* is to *pe* in the ſame ratio, and the point *p*, Cor. Prop. 17. is in the curve.

Cor. Hence it is evident, that if any line be drawn through the center of an hyperbola within the angle contained between the aſymptotes it will meet both the curves.

PROP. XXI.

The diſtance between the focus and the point in which the aſymptote cuts the directrix, as alſo the tangent at the vertex, intercepted between the vertex and aſymptote, are each of them equal to half the conjugate axis.

FOR

FIG. 18. FOR the ſquare of SH is equal to the difference of the ſquares of SC, CH, or to the difference of the ſquares of SC, CA; which, Prop. 8. is equal to the ſquare of BC; therefore SH is equal to BC. Secondly, becauſe CH is equal to CA, and the angles SHC, CAa, are right angles, and the angle SCH is common to the two triangles SHC, CAa, the triangles are equal, and Aa is equal to SH, which was proved equal to BC.

COR. If Aa and AG be taken in the tangent GAa each of them equal to CB, and the lines Ca, CG be drawn from the center, the poſition of the aſymptotes will be determined.

DEFINITIONS.

FIG. 18. XIX. If AM be the tranſverſe axis, and Bb the conjugate axis of any two oppoſite hyperbolas, and two other hyperbolas be deſcribed, of which the tranſverſe axis is Bb, and the conjugate axis AM, theſe hyperbolas are ſaid to be conjugate to the former.

XX. When the two axes are equal the hyperbolas are ſaid to be equilateral.

PROP. XXII.

The aſymptotes are diagonals of the rectangle which is made by drawing tangents through the vertices of the four hyperbolas.

FIG. 18. LET the tangents GAa, IMi, which are drawn through the vertices of the tranſverſe axis, meet the aſymptotes in G, a, and I, i. Join IB, GB, as alſo ab, ib. Then, becauſe CM is equal to CA, and

 the

the angles at A and M are right angles, and the angles at C vertical, the triangle CMI will be equal to the triangle CAa, and MI equal to Aa, which is equal to CB, by the preceding propoſition. In the ſame manner it may be proved that Mi is equal to Cb, or to CB; therefore IB, BG are equal and parallel to MC, CA, the angles IBC, GBC are each of them a right angle, and IBG is one ſtraight line, which is equal and parallel to MA. For the ſame reaſon iba is one ſtraight line, which is equal and parallel to MA; and becauſe the lines IBG, iba are perpendicular to the axis BCb, they are tangents to the conjugate hyperbolas, and $IGai$ is a rectangle, of which the aſymptotes Ia, Gi are the diagonals.

Cor. 1. The aſymptotes GCi, ICa are alſo aſymptotes to the conjugate hyperbolas. For BI and BG are each of them equal to CA, which is the ſemi-conjugate axis to the hyperbolas LBR, lbr.

Cor. 2. If the hyperbolas be equilateral, the aſymptotes will be perpendicular to each other: for CA being equal to AG, or to Aa, each of the angles ACG, ACa will be half a right angle; and therefore the angle GCa will be a right angle.

PROP. XXIII.

If a right line Pp, which cuts a conic ſection or oppoſite ſections in two points P, p, meets the directrix in H, and a right line HST be drawn through the focus, and SP, Sp be joined; the angle PSH will be equal to the angle pST. Fig. 20, 21, 22.

DRAW

DRAW pT parallel to PS, and let it meet HS in T; and draw PE, pe perpendicular to the directrix. Then the triangles HPE, Hpe will be similar, as also the triangles HSP, HTp; and SP is to PE as Sp to pe; and alternately, SP is to Sp as PE to pe, as HP to Hp, or as SP to Tp; therefore Sp is equal to Tp, and the angle pST is equal to the angle pTS, which is equal to the angle PSH.

FIG. 20, 22. COR. 1. When P and p coincide, or when HP becomes a tangent to the conic section, SP will coincide with Sp, and each of the angles PSH, pST will be a right angle.

FIG. 23, 24, 25. COR. 2. Hence, if a line SP be drawn from the focus to any point P in a conic section, and SH be drawn perpendicular to SP, meeting the directrix in H, and HP be joined, it will touch the conic section in the point P.

FIG. 20, 21, 22. COR. 3. Let the line HP, which meets the directrix in H, cut a conic section in any point P; join SP, draw HST through the focus, and make the angle TSp equal to the angle HSP; then if HP, Sp be produced till they meet in p, the point p will be in the conic section, or in the opposite section. For the angle TSp is equal to the angle HSP, or to the angle STp, and Tp is equal to Sp; and Tp, or Sp, is to SP as pH to PH, or as pe to PE; and alternately, Sp is to pe as SP to PE; therefore p is a point in the conic section, Cor. Prop. 17.

COR. 4. Any line drawn parallel to an asymptote will meet the hyperbola, or the opposite hyperbola, only in one point.

FIG. 20, 22. It is evident that it will only meet one of the hyperbolas, from Prop. 20. Therefore let HP be drawn parallel to an asymptote, meeting the hyperbola in P; and if it be possible, let it meet the

same

ſame curve in ſome other point p; and join SP, Sp. Becauſe HP is equal to SP, Cor. 2. Prop. 19. the angle SHP is equal to the angle PSH, or to the angle pST; which is impoſſible, pST being the exterior angle of the triangle pHS.

Cor. 5. It is evident, from this propoſition, that a ſtraight line cannot meet a conic ſection in more points than two.

PROP. XXIV.

If two tangents be drawn at the extremities of any line paſſing through the focus of a conic ſection, which is terminated both ways by the curve, or by the oppoſite curves, they will meet in the directrix; and they will contain a right angle in the parabola, an acute angle in the ellipſe, and an obtuſe angle, or an acute angle in the hyperbola, according as the line which paſſes through the focus is terminated by the ſame, or by the oppoſite curves.

LET PSp be any line paſſing through the focus, which is terminated by the curve, or by the oppoſite curves in P, p. From S draw SH perpendicular to PSp, meeting the directrix in H, and join HP, Hp, which will touch the curve in the points P, p, Cor. 2. Prop. 23. Draw PE, pe perpendicular to the directrix; and SP, PE will be the ſines of the angles SHP, PHE, HP being made radius, and Sp, pe the ſines of the angles SHp, pHe, Hp being radius. But SP is equal to PE in the para-

Fig. 23, 24, 25, 26.

FIG. 24. parabola, and Sp is equal to pe; therefore the angle SHP is equal to the angle PHE, and the angle SHp equal to the angle pHe; and the angle PHp is equal to the sum of the angles PHE, pHe; therefore it is a right angle. FIG. 23. If the section be an ellipse, SP will be less than PE; therefore the angle SHP will be less than the angle PHE, and the angle SHp less than the angle pHe, and the whole angle PHp less than the sum of the angles PHE, pHe, and consequently less than a right angle. FIG. 25. If the line PSp be terminated by the same hyperbola, SP will be greater than PE, and Sp greater than pe; therefore the angle PHp will be greater than the sum of the angles PHE, pHe, and consequently greater than a right angle. FIG. 26. But if the line SPp be terminated by the opposite curves, the angle SHp, in the right angled triangle HSp, is less than a right angle; and therefore the angle PHp is less than a right angle.

PROP. XXV.

If a tangent be drawn to any point in the Parabola, it will bisect the angle contained between two right lines drawn from the point of contact, one to the focus, and the other perpendicular to the directrix.

FIG. 24. LET the line PH, which touches the parabola in any point P, meet the directrix in H. Join SP, SH; and draw PE perpendicular to the directrix. The angle SPE is bisected by the line PH. For the angle PHS is equal to the angle PHE, the angles PSH, PEH are right angles, and PH is common to the two triangles PSH, PEH; therefore

fore the triangles are equal; and the angle *SPH* is equal to the angle *EPH*.

COR. 1. Hence, if the right line *PH* bisects the angle *SPE*, it will be a tangent to the parabola in the point *P*.

COR. 2. If the tangent be produced till it meet the axis in *T*, the segment of the axis intercepted between the focus and the tangent will be equal to the distance of the focus from the point of contact. For the angle *STP* is equal to the alternate angle *TPE*, which is equal to the angle *SPT*; therefore *ST* is equal to *SP*. PL. VII. FIG. 51.

PROP. XXVI.

If two tangents *PH*, *pH* be drawn at the extremities of any line which passes the focus of the Parabola, and a right line *HI* be drawn from the point of concourse parallel to the axis, it will bisect the line *Pp* in *I*: and *HI* will be bisected by the curve in the point *A*. FIG. 24.

FOR the angle *IHP* is equal to the alternate angle *HPE*, which is equal to the angle *HPI*, by the preceding proposition; therefore *IP* is equal to *IH*; and for the same reason *Ip* is equal to *IH*; and therefore *IP* is equal to *Ip*. Secondly, *SA* being equal to *AH*, the angle *ASH* is equal to *AHS*; but the angle *SHI* and *SIH* are together equal to a right angle, and therefore equal to *ASH* and *ASI*; and if from these be taken the equal angles *SHI*, *ASH*, the remaining angles *AIS*, *ASI* will be equal, and *AI* will be equal to *AS*, which is equal to *AH*.

PROP. XXVII.

If a tangent be drawn to any point in an Ellipſe or an Hyperbola, and two lines be drawn from the point of contact to the foci; the angles contained between each of theſe lines and the tangent are equal.

Fig. 27, 28. LET the line *PT* touch the ellipſe or hyperbola in any point *P*, and let it meet the directrices in *T* and *t*. Through *P* draw *EPe*, Fig. 27. and *PEe*, Fig. 28. parallel to the axis *AM*, meeting the directrices in *E* and *e*, which will be perpendicular to the directrices. Draw *PS*, *PH* to the foci, and join *ST*, *Ht*. Becauſe the triangles *TPE*, *tPe* are ſimilar, *PE* is to *PT* as *Pe* to *Pt*, and

SP is to *PE* as *HP* to *Pe*; therefore

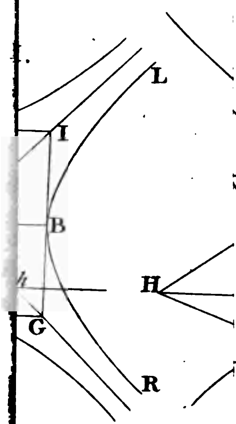
SP is to *PT* as *HP* to *Pt*, and the angles *PST*, *PHt* are right angles, Cor. 1. Prop. 23. therefore the triangles *SPT*, *HPt* are ſimilar, and the angle *SPT* is equal to the angle *HPt*.

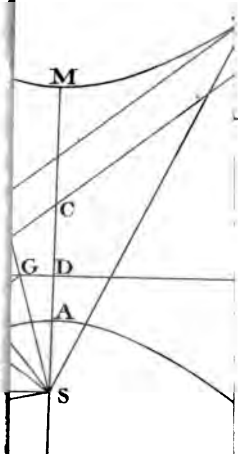
Cor. 1. If a line *TP* be drawn through any point *P* in the ellipſe or hyperbola, biſecting the angle *SPH* in the latter, and its ſupplement in the former, it will touch the curve in the point *P*.

Cor. 2. If a line *PR* be drawn from any point *P* perpendicular to the tangent, meeting the axis in *R*; it will biſect the angle contained between the lines which are drawn from the point of contact to the foci in the ellipſe, and the angle which is contained between one of theſe lines and the other produced in the hyperbola. For the angle *RPT* being equal to the angle *RPt*, Fig. 27. and equal to *RPW*, Fig. 28. if from each of theſe be taken *SPT*,

HPt,

Pl. 3.
Page 36.





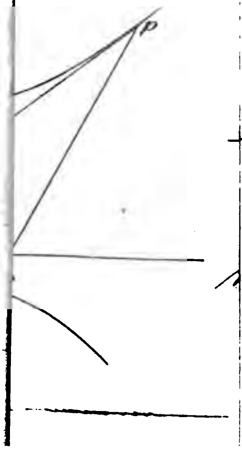


FIG. XXI.

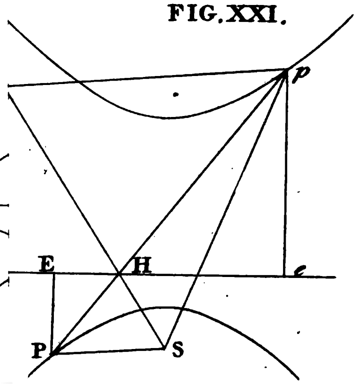


FIG. XXIII.

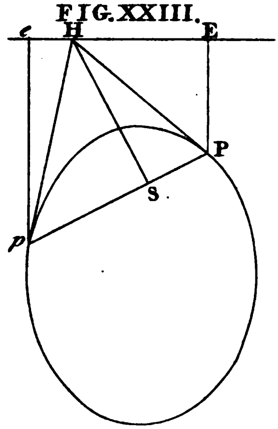
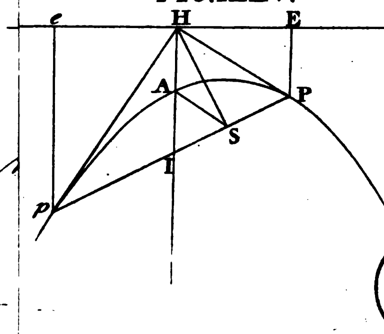


FIG. XXIV.



HPt, Fig. 27. and *SPT*, *hPW*, Fig. 28. the remaining angles *RPS*, *RPH* in the ellipse, and *RPS*, *RPh* in the hyperbola will be equal.

PROP. XXVIII.

The tangents at the vertices of any diameter of an Ellipse or an Hyperbola are parallel.

LET *PCG* be any diameter of the ellipse or hyperbola. Draw the tangents *PQ*, *GR*; and join *SP*, *PH* and *SG*, *GH*. Then, because *SC* is equal to *CH*, *CP* equal to *CG*, and the angles at *C* are vertical, the triangles *SCP*, *HCG* are equal, and the angle *SPC* is equal to the angle *HGC*, and *SP* is equal and parallel to *GH*; therefore *PH* is equal and parallel to *SG*, and *SPHG* is a parallelogram; therefore the angle *SPH* is equal to the angle *SGH*; and the halves of these angles, Fig. 30. and the halves of their supplements, Fig. 29. will be equal, that is, the angle *SPQ* equal to the angle *HGR*; and if these be added to the equal angles *SPC*, *CGH* in the ellipse, and substracted from them in the hyperbola, *CPQ* will be equal to *CGR*; and therefore *PQ* is parallel to *GR*. FIG. 29. 30.

COR. If two tangents be drawn at the vertices of any other diameter of the ellipse, or of any diameter of the conjugate hyperbolas, which are produced to meet the tangents *PQ*, *GR*, a parallelogram will be formed, which is circumscribed about the ellipse, and inscribed in the four hyperbolas.

DEFINITIONS.

Fig. 27, 28. XXI. The right line *PR*, which is drawn from the point of contact perpendicular to the tangent, intercepted between the tangent and the axis of a conic section, is called a Normal.

Fig. 28. XXII. The segment of the axis *NR*, which is intercepted between the ordinate and the normal, is called a Subnormal.

XXIII. If a right line be drawn through any point in the diameter of a conic section parallel to the tangent at its vertex, which is terminated both ways by the curve, it is called an Ordinate to that diameter.

XXIV. The segment of any diameter of a conic section, which is intercepted between an ordinate and the vertex, is called an Abscissa.

XXV. A diameter which is parallel to the tangent at the vertex of any diameter of the ellipse or hyperbola, is called a Conjugate Diameter.

XXVI. A line which is a third proportional to any diameter of the ellipse or hyperbola and its conjugate, is called a Parameter to that diameter.

XXVII. If a line be drawn through the focus of a parabola parallel to the ordinates of any diameter, which is terminated both ways by the curve, it is called a Parameter to that diameter.

PROP. XXIX. PROB. V.

To draw a tangent to a conic section from any given point without, which is not the center of the hyperbola.

IF

IF the given point H be in the directrix; draw HS to the focus which is nearest to the directrix; draw SP perpendicular to SH, meeting the curve in P, and join HP, which will touch the conic section in P, Cor. 1. Prop. 23. FIG. 31, 32.

If the given point be in any other situation, as at L; join LS, and draw LX perpendicular to the directrix. Take LD to LX in the determining ratio, and from the center L, at the distances LD, describe a circle DMq; and because LS is to LX in a greater ratio than LD to LX, Prop. 17. LS is greater than LD, and the point S is without the circle. From S draw SQ a tangent to the circle, and let it meet the directrix in H. Join LQ, and draw SP parallel to it, or perpendicular to SH. Join HL and produce it till meet SP in the point P, which is in the conic section, and the line HP touches the curve in P. For, because the triangles HQL, HSP are similar, as also the triangles LHX, PHE, SP is to PH as QL to LH, and

PH is to PE as LH to LX; therefore

SP is to PE as QL to LX, that is, in the determining ratio; therefore P is a point in the curve; and because PSH is a right angle, PH is a tangent, Cor. 1. Prop. 23.

COR. 1. Because two lines SQ, Sq may be drawn from the point S to touch the circle, two tangents LP, Lp may be drawn from the point L to the conic section.

COR. 2. The lines LP, Lp will touch the hyperbola which is on the same side of the directrix with the focus, or the opposite curve, according as SQ, Sq touch the circle on the opposite side, or on the same side of the directrix with the focus. FIG. 32.

COR. 3. If the line SQ meets the circle in the directrix, the line LHP becomes an asymptote.

For

For in that case LQ and LH would coincide; and LQ would be to LX as radius to the sine of the angle LHX; therefore LH would be inclined to the directrix at the same angle as the asymptote is; and it would never meet the curve, because SP is parallel to LQ, or LH; therefore LH coincides with the asymptote.

PROP. XXX.

Fig. 33, 34, 35. If two right lines Pp, Qq, which meet each other in any point L, and are inclined to the directrix at any given angles LHX, LhX, cut a conic section, or opposite sections, in the points P, p and Q, q; the rectangles under the segments LP, Lp and LQ, Lq will be in a constant ratio to each other, wherever the point L be taken.

LET S be the nearest focus. Join HS, and produce it if necessary; also join SP, Sp. Draw LX, PE perpendicular to the directrix; and from the point L draw LT, Lt parallel to SP, Sp, meeting the line HS in T and t. Because the angle PSH, Prop. 23. is equal to the angle pST, Fig. 33 and 35. and equal to pSW, Fig. 34. the angle LTt is equal to the angle LtT, and LT is equal to Lt. From the center L, at the distance LT, or Lt, describe a circle, cutting the line HPp, in M and m. Join SL, and produce it till it meet the circle in D and d; and because the triangles HPE, HLX are similar, as also the triangles HPS, HLT, LT is to SP as LH to PH, or as LX to PE; and alternately, LT is to

LX as SP to PE, that is, in the determining ratio; therefore the radius of the circle is the ſame when L is at the ſame diſtance from the directrix, whatever be the poſition of the line Pp. And becauſe LT is parallel to PS, and Lt parallel pS,

LP is to TS as LH is to TH, and

pL is to St as LH is to tH; therefore

Lem. 1. the rectangle PLp is to the rectangle TSt as the ſquare of LH is to the rectangle THt; but the rectangle TSt is equal to the rectangle DSd, and the rectangle THt is equal to the rectangle MHm, or to the difference of the ſquares of LH and LM; therefore the rectangle PLp is to the rectangle DSd as the ſquare of LH is to the difference of the ſquares of LH and LM; but LH is to LT, or LM, as PH to PS, and the ſquare of LH is to the ſquare of LM as the ſquare PH is to the ſquare of PS; and by diviſion, the ſquare of LH is to the difference of the ſquares of LH and LM as the ſquare of PH to the difference of the ſquares of PH and PS; which ratio depends only upon the determining ratio and the angle LHX, SP being to PH in a ratio which is compounded of the ratios of SP to PE, and PE to PH, or of the determining ratio, and the ſine of the angle LHX to radius. In the ſame manner it may be proved, that the rectangle QLq is to the rectangle DSd in a ratio which depends only on the determining ratio and the angle LhX; therefore the rectangle PLp is to the rectangle QLq in a conſtant ratio, whatever be the diſtance of the point L from the directrix.

Cor. 1. If either of the lines Pp, Qq, or both of them become tangents to the conic ſection, or oppoſite ſections, the ſquares of the tangents muſt be ſubſtituted for the rectangles PLp, QLq. For Fig. 31, let LP touch the conic ſection in P. Then, QL 32.

 being

being parallel to SP, by the preceding propoſition, LP is to QS as LH to QH; and the ſquare of LP, Lem. 1. is to the ſquare of QS as the ſquare of LH is to the ſquare QH; but the ſquare of QS is equal to the rectangle DSd, and the ſquare of QH is equal to the rectangle MHm; therefore the ſquare of LP is to the rectangle DSd as the ſquare of LH is to the rectangle MHm, or to the difference of the ſquares of LH, LM, which was proved to be a conſtant ratio; therefore the ſquare of LP is to the ſquare of Lp, or to the rectangle QLq, Fig. 34. in a conſtant ratio.

Cor. 2. If the determining ratio be that of a to b, and the ſines of the angles contained between each of the lines Pp, Qq and the directrix be R and S, the radius being unity, the ratio of the rectangles PLp and QLq will be that of $b^2-a^2R^2$ to $b^2-a^2S^2$.

For SP is to PE as a to b, and

PE is to PH as R to 1; therefore

SP is to PH as aR to b, and the ſquare of SP is to the ſquare of PH as a^2R^2 to b^2, and the ſquare of PH is to the difference of the ſquares of PH and SP as b^2 to $b^2-a^2R^2$: therefore

$LP\times Lp : SD\times Sd :: b^2 : b^2-a^2R^2$, and

$SD\times Sd : LQ\times Lq :: b^2-a^2S^2 : b^2$; therefore

$PL\times Lp : LQ\times Lq :: b^2-a^2S^2 : b^2-a^2R^2$.

Cor. 3. If the focus, the directrix, and the determining ratio be given, and a right line be given in poſition; the points in which it meets the conic ſection, or oppoſite ſections, may be found. If the right line paſſes through the focus, the points of interſection may be found by Prop. 18. and if it be parallel to the directrix, by the fourth propoſition. Therefore let the right line LH, which does not paſs through the focus, meet the directrix in H. Take any point L in that line; draw LS to the focus,

Fig. 33, 34, 35.

cus, and LX perpendicular to the directrix. In LS, produced if neceſſary, take LD to LX in the determining ratio; and from the center L, at the diſtance LD, deſcribe a circle; and join HS. Then it is evident, from this and the preceding propoſition, that the line HS will cut the circle in two points T, t, or touch it, according as the line HL cuts the conic ſection, or touches it. Therefore join LT, Lt; and draw SP, Sp parallel to LT, Lt, meeting the line HL in P, p, which are points in the ſection, or ſections. For SP is to TL as PH is to LH, or as PE to LX; and alternately, SP is to PE as TL to LX, that is, in the determining ratio; therefore P is in the curve; and for the ſame reaſon p is a point in the curve.

PROP. XXXI.

If two right lines QL, Pp, meeting each other in any point L, one of which is parallel to the axis, and the other is inclined to the directrix at any given angle, cut a parabola in the points Q, P and p; the rectangle under the ſegment QL and the latus rectum will be to the rectangle under the ſegments LP, Lp in a conſtant ratio, wherever the point L be taken. FIG. 36.

DRAW LX perpendicular to the directrix; and from the center L, at the diſtance LX, deſcribe a circle. Join QS, XS, and let XS, produced if neceſſary, meet the circle in T, and join LT. Draw SO perpendicular to LX; take OI equal to

 OX,

OX, and join *SI*, which will be equal to *SX*. Then, *LT* being equal to *LX*, and *QS* equal to *QX*, *LT* is to *LX* as *QS* is to *QX*; therefore *LT* is parallel to *QS*; and becauſe the angle *QSX* is equal to the angle *SXQ*, which is equal to the angle *SIX*, the triangles *QXS*, *SXI* are ſimilar, and *IX* is to *XS*, as *XS* to *SQ*, or *XQ*, or as *ST* to *QL*; therefore the rectangle under *IX*, *QL* is equal to the rectangle *XST*, or to the rectangle *DSd*; which, by the preceding propoſition, is to the rectangle *PLp* in a conſtant ratio; but *IX* being equal to twice *OX* the diſtance of the focus from the directrix, it is equal to the latus rectum; therefore the rectangle under *QL* and the latus rectum is to the rectangle *PLp* in a conſtant ratio.

Cor. 1. Hence the rectangle under *QL* and any other conſtant quantity is to the rectangle *PLp* in a conſtant ratio.

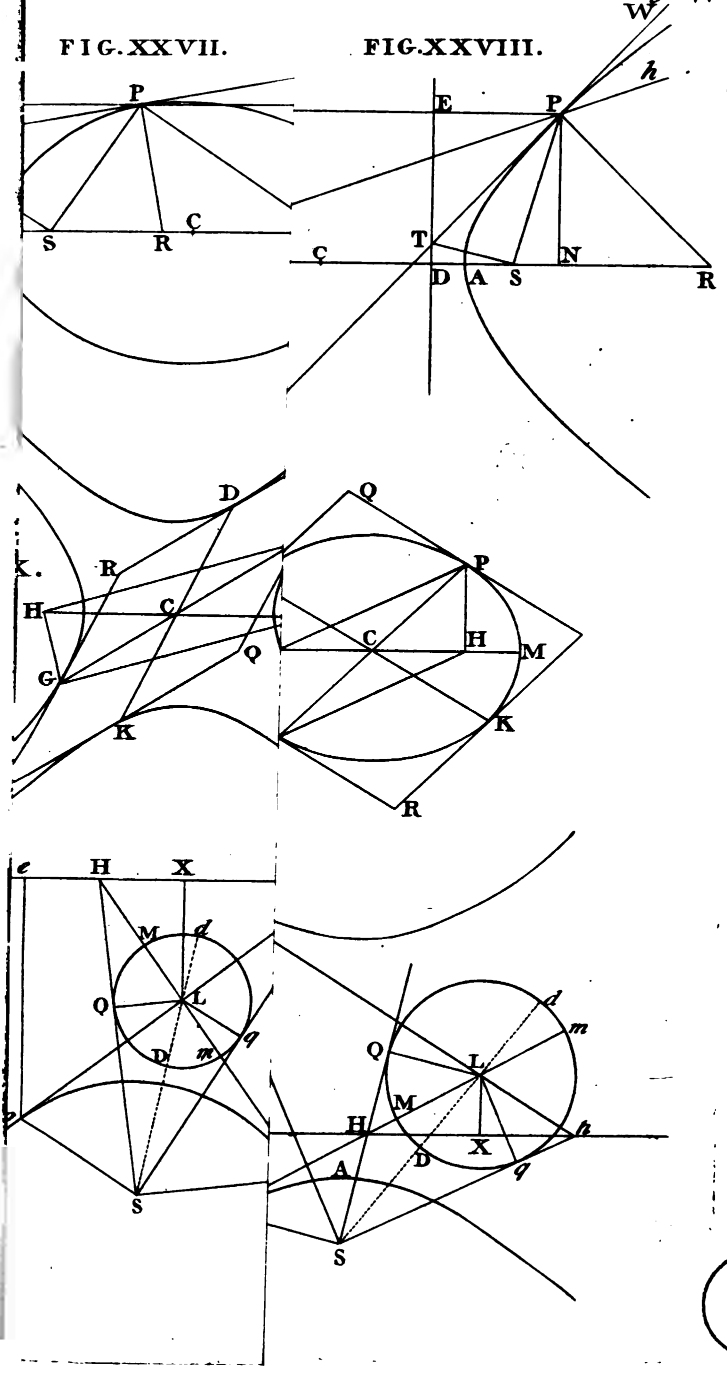
Fig. 31. Cor. 2. If the line *LP* becomes a tangent to the parabola; the ſquare of *LP* will be to the rectangle under the latus rectum and the ſegment of the diameter intercepted between the point *L* and the vertex in the ſame conſtant ratio.

LEMMA II.

Fig. 37. If a ſtraight line *AB* be divided in two points *C* and *D* in ſuch a manner, that the rectangle *CAD* ſhall be equal to the rectangle *DBC*, or the rectangle *ACB* equal to the rectangle *BDA*, the part *AC* will be equal to the part *BD*.

FIRST,

FIG. XXVII.   FIG. XXVIII.



FIRST, let the rectangle CAD be equal to the rectangle DBC. Bisect CD in E; and the rectangle CAD, together with the square of EC is equal to the square of AE, and the rectangle DBC, together with the square of ED, is equal to the square of BE; but the square of ED is equal to the square of EC; therefore the square of BE is equal to the square of AE, and BE is equal to AE; and therefore the part BD is equal to the part AC.

Secondly, let the rectangle ACB be equal to the rectangle BDA. Bisect the line AB in E; and the rectangle ACB, together with the square of EC, is equal to the square of AE; and the rectangle BDA, together with the square ED, is equal to the squaie of BE; but the square of BE is equal to the square of AE; therefore the square of ED is equal to the square of EC, and ED is equal to EC; and therefore BD is equal to AC.

PROP. XXXII.

All right lines drawn parallel to any diameter of the ellipse or hyperbola, which are terminated both ways by the ellipse or opposite hyperbolas, are bisected by the conjugate diameter.

LET ACB be any diameter of the ellipse or hybola. Through the vertices A and B draw the tangents AL, BM; and through the center C draw the diameter DCK parallel to AL, or BM, which will be the conjugate diameter. Through any point N, in the diameter DCK, draw LNM parallel to AB, meeting the ellipse or the opposite hyper- FIG. 38, 39.

hyperbolas in the points *P*, *Q*, and the tangents *AL*, *BM* in *L* and *M*. Then, *AL* being parallel to *CN* and *BM*, and *LNM* parallel to *ACB*, *AL* will be equal to *BM*, and *LN* equal *NM*; and, Cor. 1. Prop. 30. the ſquare of *LA* is to the rectangle *PLQ* as the ſquare of *MB* is to the rectangle *QMP*; but the ſquare of *LA* is equal to the ſquare of *MB*; therefore the rectangle *PLQ* is equal to the rectangle *QMP*; and therefore, Lem. 2. *PL* is equal to *QM*; and if theſe be taken from the equal lines *LN*, *MN*, Fig. 38. and added to them, Fig. 39. *PN* will be equal to *NQ*.

Cor. 1. If the diameter *DK* biſect all lines drawn parallel to *AB*, it will be the conjugate diameter to *AB*.

Fig. 38. Cor. 2. If a right line *RDT* be drawn through *D* the vertex of the conjugate diameter parallel to *AB*, it will touch the ellipſe in the point *D*. For if not, let it meet the curve in ſome other point *d*, and *Rd* will be equal to *TD*; but *TD* is equal to *RD*; therefore *Rd* is equal to *RD*, which is abſurd.

Cor. 3. Hence, if the diameter *DK* be conjugate to any diameter *AB*, *AB* will alſo be conjugate to *DK*.

PROP. XXXIII.

Every diameter of a conic ſection biſects all its ordinates.

FIRST, if the conic ſection be an ellipſe, it is evident from the preceding propoſition: for the ordinates of any diameter are parallel to the conjugate diameter.

Secondly,

Secondly, if the ſection be an hyperbola, of which *ACB* is any diameter; in the tangent *LAR* take *AR* equal to *AL*. Through *L* and *R* draw the lines *PLQ*, *FRG* parallel to *AB*, meeting the oppoſite hyperbolas in *P*, *Q* and *F*, *G*, and the tangent at the vertex *B* in *M* and *T*. Join *PF*, cutting the diameter in *V*. Then *PF* will be an ordinate which is biſected in *V*: for *PL* is equal to *MQ*, by the preceding propoſition, and *FR* equal to *TG*; and the rectangle *FRG* is to the ſquare of *RA* as the rectangle *PLQ* is to the ſquare of *LA*, Cor. 1. Prop. 30. but the ſquare of *RA* is equal to the ſquare of *LA*; therefore the rectangle *FRG* is equal to the rectangle *PLQ*, that is, the rectangle *RFT* is equal to the rectangle *LPM*; therefore Lem. 2. *RF* is equal to *PL*, and *PLRF* is a parallelogram; and therefore *PF* is parallel to the tangent *LAR*, and *PV* is equal to *VF*. FIG. 39.

Laſtly, let the ſection be a parabola, of which *AN* is any diameter, and *PNQ* an ordinate. Through the vertex *A* draw the tangent *LAM*; and draw *PL*, *QM* parallel to *NA*. Then *PLMQ* is a parallelogram, and *QM* is equal to *PL*; but the rectangle under *LP* and the latus rectum is to the ſquare of *LA* as the rectangle under *MQ* and the latus rectum is to the ſquare of *MA*, Cor. 2. Prop. 31. and the two rectangles being equal, the ſquare of *MA* is equal to the ſquare of *LA*, and *MA* is equal to to *LA*; and therefore *QN* is equal to *PN*. FIG. 40.

COR. 1. If a right line *PQ*, which is terminated by a conic ſection in the points *P*, *Q*, and which does not paſs through the center of the ellipſe, be biſected by any diameter, it is parallel to the tangent at the vertex of that diameter; for if it be not parallel to this tangent, let it be parallel to the tan- FIG. 41.

e gent

gent at the vertex of ſome other diameter; then PQ would alſo be biſected by this diameter, which is abſurd.

Cor. 2. Two right lines terminated by a conic ſection, which do not paſs through the center of the ellipſe, cannot biſect each other: for if it be poſſible, let the two lines DB, RK biſect each other in C, and let CA be the diameter which paſſes through C; then both the lines will be parallel to the tangent at the vertex A; and therefore they are parallel to each other, which is abſurd.

Cor. 3. A right line biſecting two parallel right lines, which are terminated by a conic ſection, is a diameter: for a diameter which biſects one of them will biſect the other.

Cor. 4. Hence, if a ſegment of a conic ſection be given, the diameters and center of the ſection may be found: for let two parallel right lines be drawn which are terminated by the ſegment, and the right line which biſects them will be a diameter; in the ſame manner any other diameter may be found; and if it be parallel to the former, the ſection will be a parabola; but if the diameters cut each other, the point of interſection will be the center of the ellipſe or hyperbola.

Cor. 5. If a right line CN, which biſects the two parallels DB, PQ, be produced till it meet the curve in A, and through the point A the right line IAG be drawn parallel to PQ, it will touch the conic ſection in A.

Cor. 6. Hence we have a method of drawing a tangent to a conic ſection, which ſhall be parallel to any line which cuts the ſection in two points. Let PQ be any line cutting the ſection in two points P, Q; draw any other line DB parallel to PQ; biſect PQ, DB in N and C; join CN, and pro-

produce it till it meet the conic ſection in *A*, and through the point *A* draw *IAG* parallel to *PQ*, which will be the tangent required.

PROP. XXXIV.

If two tangents be drawn at the extremities of any right line which is terminated by a conic ſection, and which does not paſs through the center of the ellipſe, they will meet each other in the diameter which biſects that right line.

LET *PQ* be the right line which is terminated by a conic ſection in *P* and *Q*. Biſect *PQ* in *N*, and through *N* draw the diameter *CNT*. Through the point *P* draw the tangent *PT* meeting the diameter in *T*, and join *TQ*, which will touch the ſection in *Q*. For draw any other line *DCB* parallel to *PNQ*, meeting the lines *TP*, *TQ* in *L* and *M*. Becauſe the triangles *TNP*, *TCL* are ſimilar, as alſo the triangles *TNQ*, *TCM*, *TN* is to *NP* as *TC* is to *CL*; and alternately, *TN* is to *TC* as *NP* is to *CL*; but *TN* is to *TC* as *NQ* is to *CM*; therefore *NP* is to *CL* as *NQ* is to *CM*; and alternately, *NP* is to *NQ* as *CL* is to *CM*; therefore *CM* is equal to *CL*, which is greater than *CD*, or *CB*; and therefore the point *M* is without the conic ſection, and the line *TQ* meets the curve only in one point *Q*. FIG. 41.

COR. 1. The tangent *IAG* at the vertex of the diameter *CA*, which is terminated by the two tangents *TP*, *TQ*, is biſected in *A*.

COR. 2. If two right lines which touch a conic

 ſection

section meet each other; a right line drawn from their point of concourse bisecting the line which joins the points of contact will be a diameter of the section.

PROP. XXXV.

If a right line cutting the hyperbola, or the opposite hyperbolas, meets the asymptotes in two points; the segments between the hyperbola or hyperbolas and asymptotes will be equal.

FIG. 42. LET the line PQ cut the hyperbola, or the opposite hyperbolas in the points P, Q, and meet the asymptotes in R, T; the segments PR, QT will be equal. If PR be not equal to QT, let one of them, as QT, be the greater; and cut off the part QO equal to PR; join CO, which being produced will meet the hyperbola in some point q, Cor. Prop. 20. Through the point q draw qpr parallel to QP meeting the curve in p and the asymptote in r. Bisect PQ in N, and draw the diameter CNn, and PQ, pq will be ordinates of that diameter. Then, NQ being equal to NP, and QO equal to PR, NO will be equal to NR; and ON is to qn as CN to Cn, as NR is to nr; therefore nq is equal to nr; but nq is equal to np; therefore np is equal to nr, which is absurd; and therefore QT is not greater than PR.

COR. 1. If the line TNR be supposed to move from N to A, the points P, Q will coincide in A; and IA will be equal to AG; therefore when a line touches an hyperbola, the segments between the point of contact and the asymptotes are equal.

COR.

Pl.5.
Page 50.

FIG. XX

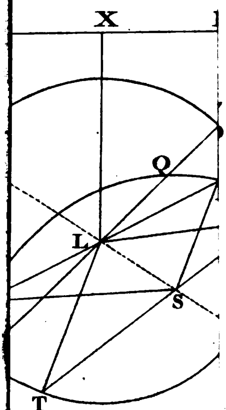


FIG. XXXVI.

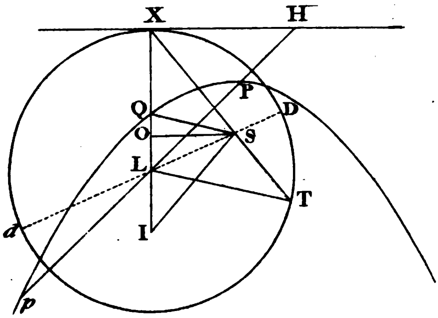


FIG. XXXIX.

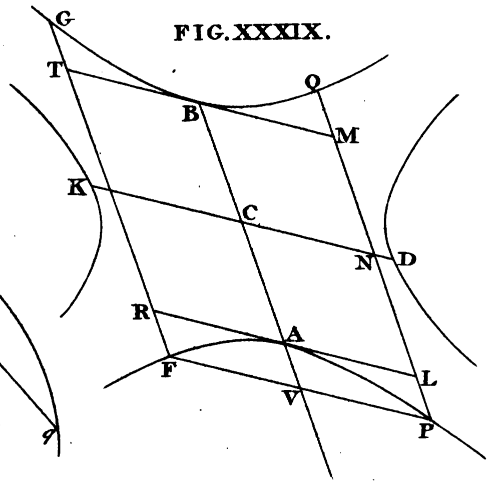


FIG. XXX

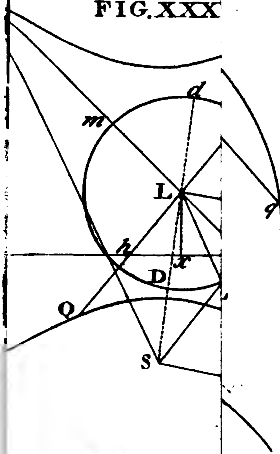


FIG. XLI.

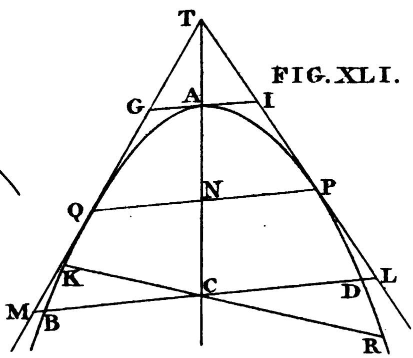


FIG. XXXV

Cor. 2. If a right line *IAG*, meeting the asymptotes in *I*, *G* and the hyperbola in *A*, be bisected in the point *A*, it will touch the hyperbola; if not, let it meet the curve in some other point *a*; and *IA* will be equal to *Ga*, and *GA* equal to *Ga*, which is absurd.

Cor. 3. Because *RP* is equal to *QT*, *RQ* is equal to *TP*; therefore the four rectangles *PRQ*, *RPT*, *QTP*, *TQR* are all equal.

PROP. XXXVI.

If two right lines be drawn from any point in an hyperbola to the asymptotes, and from any other point, in the same or opposite hyperbola, two other right lines be drawn to the asymptotes parallel to the two former; the rectangle under the first two lines will be equal to the rectangle under the other two.

FROM any point *P* in the hyperbola *PQ* draw the lines *PL*, *PH* to the asymptotes; and from any other point *Q*, in the same or in the opposite curve, draw *QE*, *QF* parallel to *PL*, *PH*. The rectangle under *QE*, *QF* will be equal to the rectangle under *PL*, *PH*. Join *PQ*, and let it meet the asymptotes in *R* and *T*: and because the triangles *TQF*, *TPH* are similar, as also the triangles *RPL*, *RQE*, *QF* is to *PH* as *TQ* is to *TP*, or as *RP* is to *RQ*, that is, as *PL* to *QE*; therefore the rectangle under *QF*, *QE* is equal to the rectangle under *PH*, *PL*. Fig. 43.

Cor. 1. Hence, if from two points *P*, *Q*, in the same

ſame or in the oppoſite hyperbolas, two right lines *PL*, *QE* be drawn to the ſame or to different aſymptotes parallel to the other aſymptote; the rectangle *CLP* will be equal to the rectangle *GEQ*: for if the parallelograms *CLPH*, *CEQF* be completed, the rectangles *HPL*, *FQE*, that is, the rectangles *CLP*, *CEQ* will be equal.

Cor. 2. Becauſe the rectangles are equal, *CL* is to *CE* as *EQ* is to *LP*, but the parallelograms *CQ*, *CP* are equiangular, therefore they are equal.

Cor. 3. Hence, if *CP*, *CQ* be joined, the triangles *CQF*, *CQE*, *CPH* and *CPL* will be equal.

PROP. XXXVII.

All right lines drawn parallel to an aſymptote, which are terminated by two conjugate hyperbolas, are biſected by the other aſymptote.

Pl. viii. Fig. 64. FROM any point *P* in the hyperbola *AP* draw *PD* parallel to the aſymptote *HC*, and let it meet the conjugate hyperbola *BD* in *D*. *PD* is biſected by the other aſymptote *CV* in *l*. Join *CP*, *CD*. Let *ACM*, *BCb* be the two axes; and join *AB* cutting the aſymptote in *T*. Through *A* draw the tangent *IAa* meeting the aſymptotes in *I* and *a*. Then *Aa* being equal and parallel to *BC*, Prop. 21. *AB* is equal and parallel to *aC*, and *IA* is to *Ia* as *AT* is to *aC*, or *AB*, but *IA* is half of *Ia*, therefore *AT* is half of *AB*; and, Cor. 1. preceding propoſition, the rectangle *ClP* is equal to the rectangle *CTA*, or to the rectangle *CTB*, which is equal to the rectangle *ClD*; therefore *lP* is equal to *lD*.

PROP.

PROP. XXXVIII.

The tangent at the vertex of any diameter of an hyperbola, which is terminated by the asymptotes, is equal to the conjugate diameter.

LET *PCG* be any diameter of the hyperbola; from *P* draw *PD* parallel to the asymptote *HC*, meeting the conjugate hyperbola in *D*; and join *CD*. Through *P* draw the tangent *VPH* meeting the asymptotes in *V* and *H*. *VP* is equal to *PH*, Cor. 1. Prop. 35. and *VP* is to *PH* as *PI* is to *HC*; therefore *PI* is half of *HC*, but it is also half of *PD*, by the preceding proposition, therefore *PD* is equal to *HC*, and *DC* is equal and parallel to *PH*, and twice *DC*, or *DK*, is equal to *VH*. Pl. VIII. Fig. 64.

Cor. 1. If *DK* be a conjugate diameter to *PG*, *PG* is also conjugate to *DK*. Join *VD*, and produce it till it meet the asymptote in *h*. Then *DC* being equal and parallel to *PH*, or *VP*, *VD* is equal and parallel to *PC*, which is equal to *Dh*, it being the opposite side of the parallelogram *PDhC*; therefore *VDH* touches the conjugate hyperbola in *D*, Cor. 2. Prop. 35. and *PCG* is a conjugate diameter to *DCK*.

Cor. 2. Hence, if two tangents *PV*, *DV* be drawn through the vertices of any two conjugate diameters, they will meet in the asymptote; and the asymptotes are diagonals of the parallelogram which is formed by the four tangents.

Cor. 3. If the hyperbolas be equilateral, the conjugate diameters will be equal: for the angle *HCI* will be a right angle, Cor. 2. Prop. 22. it will there-

therefore be in a semicircle, of which VH is the diameter and P the center; therefore CP is equal to PH, which is equal to CD.

PROP. XXXIX.

If through any point in an asymptote a right line be drawn cutting an hyperbola or opposite hyperbolas; the rectangle under the segments, between the asymptote and the hyperbola or hyperbolas, will be equal to the square of the semidiameter which is parallel to that right line.

FIG. 44. THROUGH any point R in the asymptote draw the right line RT, cutting the hyperbola or opposite hyperbolas in the points P, Q, and the other asymptote in T. The rectangle PRQ will be equal to the square of the semidiameter which is parallel to RT. Let CD be the semidiameter parallel to the line which cuts the hyperbola PAQ, and AC the semidiameter parallel to the line which cuts the opposite hyperbolas. Take any point r in the asymptote, and through r draw rt parallel to RT, cutting the curve or the opposite curves in p, q, and the other asymptote in t. From the points P, p draw PH, PL and pF, pE parallel to the asymptotes; and because the triangles PLR, pEr are similar, as also the triangles PTH, ptF,

PR is to PL as pr is to pE, and

PT is to PH as pt is to pF; therefore,

Lem. 1. the rectangle RPT is to the rectangle LPH as the rectangle rpt is to the rectangle EpF; but the rectangle LPH is equal to the rectangle EpF,

EpF, Prop. 36. therefore the rectangle RPT is equal to the rectangle rpt, or the rectangle PRQ equal to the rectangle prq, Cor. 3. Prop. 35. and if IAG be the tangent which is parallel to RT, when P is taken at A the rectangle PRQ becomes equal to the square of IA, which is equal to the square of DC; and when P in the opposite hyperbola is at M, the rectangle RPQ becomes equal to the square of AC.

Cor. 1. The four rectangles PRQ, RPT, QTP, TQR being all equal, are each of them equal to the square of the semidiameter which is parallel to the line RT.

Cor. 2. As the point P recedes from the vertex A the line RP perpetually dereases, and will become less than any assignable quantity: for the rectangle PRQ is equal to a given square, and PQ increases without limit; therefore RP must decrease without limit. Fig. 42.

PROP. XL.

If two right lines meeting each other cut or touch a conic section, or opposite sections; the rectangles under the segments between the point of concourse and the points of intersection, or the squares of the tangents will be to each other as the squares of the semidiameters to which the lines are parallel.

If the lines be parallel to any of the diameters of the ellipse, or to any of the diameters of the opposite hyperbolas, the proposition is evident from Prop.

Prop. 30. becauſe the lines which meet each other make the ſame angles with the directrix as thoſe which paſs through the center, and the latter are biſected in the center. But if either of the lines

Fig. 46. *PLQ*, *LRT*, or both the lines *PLQ*, *NLM* be parallel to ſome of the conjugate diameters of the hyperbola; produce *QLP* till it meet the aſymptote in *G*, and through *G* draw *FGH* parallel to *LRT*, meeting the oppoſite curves in *F* and *H*. Let *CB*, *CD* and *CA* be the ſemidiameters which are parallel to *QP*, *MN* and *RT*. Then, Prop. 30. the rectangle *PLQ* is to the rectangle *RLT* as the rectangle *PGQ* is to the rectangle *FGH*, or as the ſquare of *CB* to the ſquare of *CA*, by the preceding propoſition. In the ſame manner it may be proved, that the rectangle *RLT* is to the rectangle *NLM* as the ſquare of *CA* to the ſquare of *CD*; therefore the rectangle *PLQ* is to the rectangle *NLM* as the ſquare of *CB* is to the ſquare of *CD*.

If the lines touch the conic ſection or oppoſite ſections, the ſquares of the tangents will be to each other as the rectangles under the ſegments of any two lines drawn parallel to them, which meet each other, and cut the ſection or oppoſite ſections; and therefore they are as the ſquares of the ſemidiameters to which they are parallel.

Fig. 45, 46. Cor. If two right lines *IQ*, *IN*, meeting each other in *I*, touch an ellipſe or hyperbola in *Q*, *N*, and are parallel to two other lines *VT*, *VN* which meet each other in *V*, and touch the ellipſe or oppoſite hyperbola or hyperbolas in *T*, *N*; *IQ* will be to *IN* as *VT* is to *VN*: for the ſquares of *IQ*, *IN* are to each other as the ſemidiameters to which they are parallel, and the ſquares of *VT*, *VN* are in the ſame ratio.

PROP.

PROP. XLI.

If an ordinate be drawn to any diameter of an Ellipse or an Hyperbola; the rectangle under the abscissæ will be to the square of the semi-ordinate as the square of the diameter is to the square of its conjugate.

LET *ACM* be any diameter of an ellipse or hyperbola, to which *PNQ* is an ordinate; and let *DCK* be the conjugate diameter, which is parallel to *PNQ*. Then, by the preceding proposition, the rectangle *ANM* is to the rectangle *PNQ*, or the square of *PN*, as the square of *CA* to the square of *CD*, or as the square of *AM* to the square of *DK*. FIG. 47, 48.

COR. 1. Because the parameter is a third proportional to the diameter and its conjugate, the rectangle under the abscissæ is to the square of the semi-ordinate as the diameter is to the parameter.

COR. 2. The two conjugate diameters being constant, the rectangle under the abscissæ will vary as the square of the ordinate.

COR. 3. Those ordinates which are at equal distances from the center are equal; and those which are nearer to the center are greater in the ellipse, and less in the hyperbola than those which are more remote.

COR. 4. If the hyperbola be equilateral, the rectangle under the abscissæ will be equal to the square of the semi-ordinate.

PROP. XLII.

If an ordinate be drawn to any diameter of the Parabola; the ſquare of the ſemi-ordinate will be equal to the rectangle under the abſciſſa and the parameter.

Fig. 49. LET *AN* be any diameter of the parabola, to which *PNQ* is an ordinate. Draw the parameter *TSV*, cutting the diameter in *F*; join *SA*, and let the diameter be produced till it meet the directrix in *D*. Becauſe *TF* is half of *TV*, Prop. 33. aud *AF* is half of *DF*, or half of *TF*, Prop. 26. *AF* is to *TF* as *TF* is to *TV*, and the rectangle under *AF* and *TV* is equal to the ſquare of *TF*; but the rectangle under *AF* and *TV* is to the rectangle under *AN* and *TV* as the ſquare of *TF* is to the ſquare of *PN*, Cor. 1. Prop. 31. therefore the rectangle under *AN* and *TV* is equal to the ſquare of *PN*.

Cor. 1. Becauſe *TV* is equal to twice *TF*, or to four times *SA*, the rectangle under *AN* and four times *SA* is equal to the ſquare *PN*.

Cor. 2. If the tangent at the vertex of any diameter meets another diameter produced, the ſquare of the tangent will be equal to the rectangle under that part of the diameter which is produced and the parameter which belongs to the firſt diameter: for *NI* is a parallelogram, and *AI*, *IQ* are equal to *QN*, *AN*; therefore the ſquare of *AI* is equal to the rectangle under *IQ* and *TV*.

Cor. 3. The parameter being conſtant, the ſquare of the ordinate varies as the abſciſſa.

PROP.

PROP. XLIII.

If AR be taken in the tangent at the vertex of any diameter of a conic section equal to the parameter of that diameter, and a line be drawn from R to the other vertex in the ellipse and hyperbola, which line is parallel to the diameter in the parabola; and if the ordinate QNP meets the line RM in L; the rectangle under the abscissa AN and the line NL will be equal to the square of the semi-ordinate PN. FIG. 47, 48, 49.

If the conic section be a parabola, the proposition is manifest. If the section be an ellipse or an hyperbola, the rectangle ANM is to the square of PN as AM is to AR; as NM to NL, or as the rectangle ANM to the rectangle AN, NL; therefore the square of PN is equal to the rectangle AN, NL.

Cor. If the diameter of an ellipse or an hyperbola becomes infinite, and the abscissa AN be finite, RL will be parallel to AN; therefore NL will be equal to AR, and the ordinate PNQ will be equal to the ordinate of a parabola, of which AN is the diameter and AR the parameter of that diameter. Hence the ellipse or hyperbola will have the same properties as the parabola at all finite distances from the vertex.

PROP. XLIV. PROB. VI.

Two right lines being given, one of which is bisected by the other; to describe a parabola, of which the right line bisected shall be an ordinate, and the other line the diameter.

Fig. 49. LET AN, PQ be the two given lines one of which is bisected in N. Through A draw IAR parallel to QNP, and take AR a third proportional to AN and PN; produce NA to D, so that AD may be a fourth part of AR. Through the point D draw DX perpendicular to DN. Let the angle DAI be less than the angle DAR; make the angle IAS equal to the angle IAD, and take AS equal to AD. Then if a parabola be described, of which S is the focus, and DX the directrix; AN will be a diameter of that parabola, and PNQ an ordinate: for SA being equal to AD, A is a point in the parabola, Cor. Prop. 17. and because the angle SAI is equal to the angle IAD, AI is a tangent to the parabola in the point A, Cor. 1. Prop. 25. therefore QNP is parallel to the ordinates of the diameter which passes through A; and PN being a mean proportional between AN and four times AS, it must be equal to the semi-ordinate, Cor. 1. Prop. 42.

Cor. 1. Hence, if the parameter of any diameter be given, and the angle which that diameter makes with its ordinates, the parabola may be described: for any abscissa being taken in the diameter, the semi-ordinate may be found, by taking a

mean

mean proportional between the abſciſſa and the given parameter.

Cor. 2. If in any figure *PAQ* all the right lines *PQ*, *TV* &c. which are inclined to the indefinite right line *AN* at a given angle, are biſected by the line *AN*, and the ſquares of *PN*, *TF*, &c. are as the ſegments *AN*, *AF*, &c. the curve *PAQ* which paſſes through the extremities of theſe lines is a parabola, of which the indefinite right line *AN* is a diameter, and the lines *PQ*, *TV*, &c. are ordi nates: for deſcribe the parabola having one of theſe lines *PQ* for an ordinate, and *AN* an abſciſſa. Then, becauſe the ſquare of *PN* is to the ſquare of *TF* as *AN* is to *AF*, *TFV* is alſo an ordinate of that parabola. In like manner it may be proved, that all the lines *PQ*, *TV*, &c. are terminated by the parabola.

Cor. 3. From this propoſition we have a method of finding two mean proportionals between two given right lines. Let *P*, *Q* be the two given lines. Deſcribe two parabolas *BAC*, *DAC*, the axes of which *AE*, *AF* are perpendicular to each other, and of which *P*, *Q* are the parameters. From the point *C*, in which the two curves interſect each other, draw *CE*, *CF* perpendicular to the axes, and they will be the proportionals required: for *P* is to *CE*, or *AF*, as *AF* is to *AE*, or *CF*, and *AF* is to *CF* as *CF* is to *Q*. Fig. 50.

PROP.

PROP. XLV.

If a right line be drawn from the focus of a parabola perpendicular to any tangent; it will be a mean proportional between the diſtance of the point of contact from the focus, and the diſtance of the focus from the vertex.

Fig. 51. LET PT touch the parabola in any point P, and let it meet the axis in T. Draw PE perpendicular to the directrix; join SE cutting the tangent in the point Y, and join AY. Then SY will be perpendicular to PT: for, SP being equal to PE, and the angle SPY equal to the angle EPY, the triangles SPY, EPY are equal; therefore SY is equal to EY, and the angle SYP is equal to the angle PYE; therefore they are each of them a right angle. And SA being equal to AD, AY is parallel to DE, and SAY is a right angle; therefore the triangles SAY, SYT are ſimilar, and ST, or SP, is to SY as SY is to SA.

Cor. 1. Becauſe the rectangle under SP, SA is equal to the ſquare of SY, and SA is conſtant in the ſame parabola, the ſquares of the perpendiculars from the focus upon the tangents will be as the diſtances of the points of contact from the focus.

Cor. 2. If PR be drawn from the point of contact perpendicular to the tangent, meeting the axis in R; SR will be equal to ST, or SP: for RP is parallel to SE, and $RPES$ is a parallelogram; therefore SR is equal to PE, or SP.

Cor.

FIG. XLII.

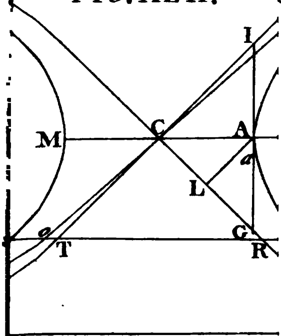


FIG. XLVI.

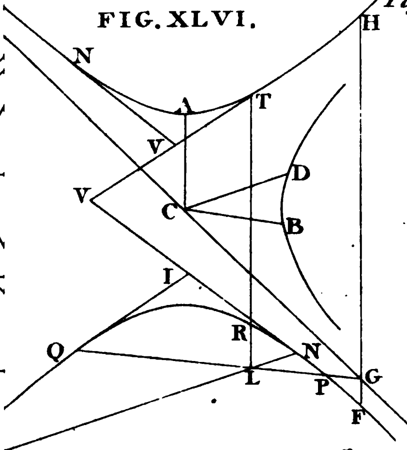


FIG. XLIII.

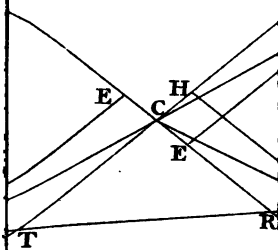
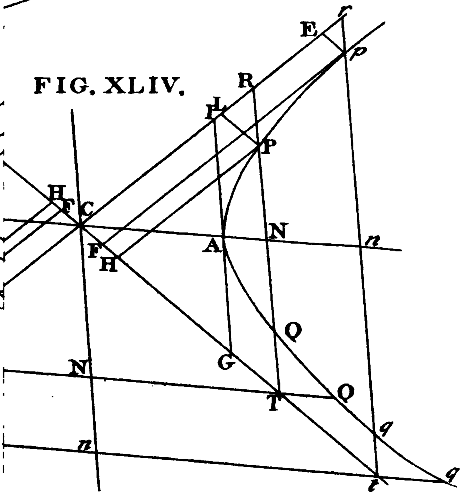
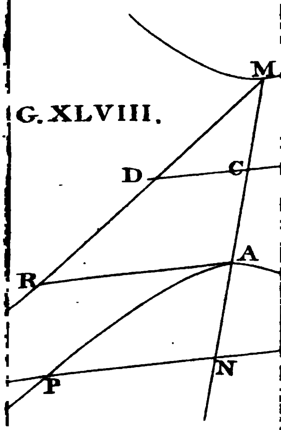


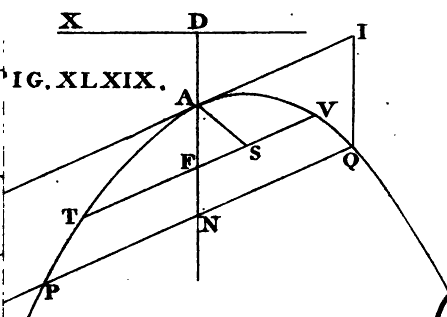
FIG. XLIV.



G. XLVIII.



IG. XLXIX.



COR. 3. The normal PR is equal to twice the perpendicular ST.

COR. 4. The ſubnormal RN is equal to half the latus rectum: for SR is equal to PE, or ND; and if from each be taken the common part SN, NR will be equal to SD, or to half the latus rectum.

PROP. XLVI.

If a tangent to any point in the parabola meets a diameter, and an ordinate be drawn to that diameter from the point of contact; the ſegment of the diameter between the vertex and the tangent will be equal to the abſciſſa.

LET TP, which touches the parabola in any point P, meet the diameter NA in T; and draw the ordinate PN. NA will be equal to AT. FIG. 52.

Through the vertex A draw the tangent AI meeting PT in I; join AP, and draw the diameter IG, cutting the line AP in F, and the ordinate PN in G. Becauſe IG is a diameter which paſſes through the point of concourſe of the two tangents PI, AI, it will biſect AP in F, Cor. 2. Prop. 34. and the triangles FIA, FGP being equiangular, AI will be equal to PG; but AI is equal to NG; therefore PG, GN are equal; and AI is half of NP; but TA is to TN as AI is to NP; therefore TA is half of TN, or TA is equal to AN.

COR. Hence we have another method of drawing a tangent to the parabola from a given point without. Let T be the given point; draw TAN parallel to the axis, cutting the parabola in A: through

through A draw the tangent AI; take AN equal to AT; and through N draw NP parallel to AI, meeting the curve in P; and join TP, which will be the tangent required.

PROP. XLVII.

If two right lines be drawn from the foci of an ellipſe or an hyperbola perpendicular to any tangent; they will meet the tangent in the circumference of a circle, which has the tranſverſe axis for a diameter.

FIG. 53, 54. LET PY touch the ellipſe or hyperbola in any point P. Join SP, PH; and draw SY, HZ perpendicular to the tangent: produce SY till it meet HP in the point W; and join CY. Becauſe the angles SPY, YPW are equal, Prop. 27. and PY is common to the two triangles SPY, YPW, PW is equal to PS, and SY equal to YW; therefore HW is equal to the tranſverſe axis, Prop. 14. and becauſe SC is equal to CH, and SY equal to YW, CY is parallel to HW; and SC is to CY as SH is to HW; but SC is half of SH; therefore CY is half of HW, or AM. And if from the center C, at the diſtance CA, a circle be deſcribed, it will paſs through the point Y. In the ſame manner it may be proved that it will paſs through Z.

COR. If the diameter DCK be parallel to the tangent at P; it will cut off from SP, PH the ſegments PE, PI, each of them equal to half the tranſverſe axis: for $CEPZ$ and $CYPI$ are parallelograms; therefore PE, PI are equal to CZ, CY, each of which is equal to CA.

PROP.

PROP. XLVIII.

The rectangle under the perpendiculars, which are drawn from the foci of an ellipse or an hyperbola to any tangent, is equal to the square of half the conjugate axis.

THE same construction remaining as in the last proposition, produce ZC till it meet YS produced in G; and let CB be the conjugate semi-axis. Because GS is parallel to HZ, the triangles CSG, CHZ are equiangular; and CS being equal to CH; SG is equal to HZ, and CG is equal to CZ, or CA; therefore the point G is in the circumference of the circle; and the rectangle GSY is equal to the rectangle ASM, that is, the rectangle under HZ, SY is equal to the square of BC. Fig. 53, 54.

Cor. 1. The square of SY is to the rectangle SY, HZ, or the square of BC, as SY is to HZ, or, because the triangles SPY, HPZ are similar, as SP to HP; therefore the square of SY is equal to

$$BC^2 \times \frac{SP}{HP}$$

Cor. 2. Hence, the square of BC being constant, the square of the perpendicular SY will vary as SP directly, and as HP inversely.

PROP. XLIX.

If a tangent to any point in an ellipse or an hyperbola meets a diameter, and from the point of contact an ordinate be drawn to that diameter; the semidiameter will be a mean proportional between the segments of the diameter, which are intercepted between the center and the ordinate, and between the center and the tangent.

Fig. 55, 56. LET the right line *PT* touch the ellipse or hyperbola in any point *P*, and let it meet the diameter *MA* in *T*; and from the point *P* draw *PNQ* an ordinate to the diameter *MA*. *CN* is to *CA* as *CA* is to *CT*. Through the vertices *A*, *M* draw the tangents *AI*, *ML*, meeting the tangent *PT* in *I* and *L*; and take *CO* on the opposite side of the center equal to *CN*. Then, Cor. Prop. 40. *IP* is to *IA* as *LP* is to *LM*; and alternately, *IP* is to *LP* as *IA* is to *LM*; and because the lines *AI*, *NP*, *ML* are parallel, *AN* is to *NM* as *TA* is to *TM*; and by division, Fig. 55. and by composition, Fig. 56. *ON* is to *AN* as *AM* is to *TA*; and by taking the halves of the antecedents, *CN* is to *AN* as *CA* is to *TA*; and by composition, Fig. 55. and by division, Fig. 56. *CA* is to *CN* as *CT* is to *CA*; and by inversion, *CN* is to *CA* as *CA* is to *CT*.

Cor. 1. The segment of the diameter intercepted between the ordinate and the center, the two abscissæ, and the segment between the ordinate and the tangent are proportionals. *CN* is to *AN* as *NM* is to *TN*; for *CN* is to *CM* as *CM* is to *CT*; and

and by composition, CN is to NM as CM is to TM; and alternately, CN is to CM as NM is to TM; and by division, CN is to AN as NM is to TN.

Cor. 2. The segments TA, TN, TC and TM are proportionals: for TC is to AC as AC is to NC; and by division, TA is to TC as AN is to AC; and alternately, TA is to AN as TC is to AC; and by composition, TN is to TA as TM is to TC; and by inversion, TA is to TN as TC is to TM.

Cor. 3. From this proposition we have another method of drawing a tangent to an ellipse or an hyperbola from a given point without. Let T be the given point; draw the diameter CT cutting the curve in A. Through the point A draw the tangent AI; take CN a third proportional to CT, CA; and draw NP parallel to AI, meeting the curve in P; and join TP, which will touch the curve in the point P.

PROP. L.

If a tangent to any point in the hyperbola meets a conjugate diameter, and an ordinate be drawn from the point of contact to that diameter; the conjugate semidiameter will be a mean proportional between the segments, which are intercepted between the ordinate and the center, and between the center and the tangent.

LET the tangent PT, which meets the diameter AM in T, be produced till it meet the conjugate diameter Bb in t. Draw the ordinate FIG. 56.

Pn parallel to AM. Becauſe CN is to CA as CA is to CT, by the preceding propoſition, the ſquare of CN is to the ſquare of CA as CN is to CT; and by diviſion, the difference of the ſquares of CN and CA is to the ſquare of CA as TN is to CT, or, becauſe PN, Ct are parallel, as PN is to Ct, or as Cn to Ct; but the ſquare of Cn is to the ſquare of CB as the rectangle ANM is to the ſquare of CA, Prop. 41. or as the difference of the ſquares of CN and CA is to the ſquare of CA; therefore the ſquare of Cn is to the ſquare of CB as Cn is to Ct; and therefore the three lines Cn, CB and Ct are proportionals, that is, Cn to CB as CB to Ct.

PROP. LI.

Fig. 55, 56. If two tangents AI, ML be drawn through the vertices of any diameter of an ellipſe or hyperbola, meeting any other tangent PT in I and L; the rectangle under the tangents AI, ML will be equal to the ſquare of the ſemidiameter CB to which they are parallel; and the rectangle under IP, PL, the ſegments of the tangent which they meet, will be equal to the ſquare of the ſemidiameter CD parallel to this tangent.

If the tangent TP be parallel to AM in the ellipſe, the propoſition is manifeſt; for each of the tangents AI, ML will be equal to the ſemidiameter CB. But if TP be not parallel to the diameter AM; let it meet it in the point T, and the diameter Bb in t. From the point P draw the ſemi-ordinates

dinates PN, Pn to the diameters AM, Bb. Then, Cor. 2. Prop. 49. TA is to TN as TC is to TM; and because the lines AI, NP, Ct and ML are parallel, AI is to NP, or Cn, as Ct is to ML; therefore the rectangle under AI, ML is equal to the rectangle under Cn, Ct, or to the square of CB, Prop. 49, and 50. Secondly,

AI is to IP as CB is to CD, Cor. Prop. 40. and ML is to PL as CB is to CD; therefore the rectangle AI, ML is to the rectangle IP, PL as the square of CB is to the square of CD; but the rectangle AI, ML is equal to the square of CB; therefore the rectangle IP, PL is equal to the square of CD.

PROP. LII.

If a tangent to any point in an ellipse or an hyperbola meets two conjugate diameters; the rectangle under the segments of the tangent, between the point of contact and the diameters, is equal to the square of the semidiameter which is parallel to the tangent.

LET the tangent PT meet the two conjugate diameters AM, Bb in T and t. From the point P draw the ordinate PNQ to the diameter AM; and through the points A, M draw the tangents AI, ML, meeting the tangent PT in I and L; and draw the diameter DCK parallel to PT. Because CN is to AN as NM is to TN, Cor. 1. Prop. 49. Pt is to PI as PL is to PT; therefore the rectangle TPt is equal to the rectangle IPL, FIG. 55. 56.

which

which is equal to the ſquare of CD, by the preceding propoſition.

Cor. Hence, if a right line TP which touches an ellipſe or hyperbola meets two diameters AM, Bb in T and t, and the rectangle TPt is equal to the ſquare of the ſemidiameter which is parallel to the tangent, AM, Bb are conjugate diameters.

PROP. LIII. PROB. VII.

Two right lines biſecting each other being given; to deſcribe an ellipſe or an hyperbola, of which the two given lines ſhall be conjugate diameters.

If the two given lines be at right angles to each other, they will be the axes; and the ellipſe or hyperbola may be deſcribed by Prop. 15. But if
Fig. 57, 58. the lines PG, DK, biſecting each other in any point C, be not at right angles to each other; through the point P draw PT parallel to DK. Take PQ in the line CP, towards the center in the hyperbola, and the contrary way in the ellipſe, a third proportional to CP, CD. Biſect CQ in the point V; and draw VR perpendicular to CQ, meeting the line PT in R. From the center R, at the diſtance RQ, or RC, deſcribe a circle cutting the line PT in T and t. Draw the lines CT, Ct, to which draw PN, Pn perpendicular. In CT take CA a mean proportional between CN and CT; and in Ct take CB a mean proportional between Cn and Ct. Make CM equal to CA, and Cb equal to CB. Then becauſe the angle TCt is in a ſemicircle, it is a right angle; and the lines AM, Bb biſect each other; therefore deſcribe an ellipſe or an hyperbola,

perbola, of which AM, Bb are the axes; and PG, DK will be conjugate diameters of that ellipse or hyperbola: for PN is to CB as CB is to Ct; therefore the square of PN is to the square of CB as PN is to Ct, that is, by similar triangles, as NT to CT; but CN being to CA as CA to CT, CN is to CT as the square of CN is to the square of CA; and by division, NT is to CT as the difference of the squares of CA and CN is to the square of CA, or as the rectangle ANM to the square of CA; therefore the square of PN is to the square of CB as the rectangle ANM is to the square of CA; and consequently PN must be a semi-ordinate, and the point P is in the curve, to which TP is a tangent, Prop. 49. and because KD is parallel to TP, and the rectangle TPt is equal to the rectangle CPQ, or to the square of CD, KD is a conjugate diameter to PG, by the preceding proposition.

Cor. 1. Hence, if any two conjugate diameters of an ellipse or an hyperbola be given, the axes may be found.

Cor. 2. If the right line PNQ drawn through any point in the line AM between the points A and M, or in MA produced, making any given angle with AM, is bisected by it in N, and the square of PN or QN is to the rectangle ANM in any given ratio; the points P, Q will be in an ellipse in the first case, and in an hyperbola in the second, of which AM is a diameter, and PQ an ordinate: for bisect AM in C; through C draw BCb parallel to PQ, and take CB, Cb such, that the square of CB, or Cb, may be to the square of CA in the given ratio; and describe an ellipse or an hyperbola, of which AM, Bb may be conjugate diameters. Then because PQ is parallel to Bb, and the square of PN is to the rectangle ANM as the Fig. 55, 56.

square

square of CB to the square of CA, PNQ is an ordinate to the diameter AM; and therefore the points P, Q are in the ellipse or hyperbola, of which AM, Bb are conjugate diameters.

PROP. LIV.

FIG. 55, 56. If a right line PT touches an ellipse or an hyperbola in any point P, and PR be drawn through the point of contact perpendicular to the tangent, meeting the two axes in R and r; the rectangle under the normals PR, Pr will be equal to the square of the semidiameter CD, which is parallel to the tangent.

LET the tangent meet the two axes in T and t. Then the triangles TPR, TCt are similar; for they have each of them a right angle, and they have a common angle at T, Fig. 55. and the angles at T are vertical, Fig. 56. and the right angled triangles TCt, rPt, which have a common angle at t, are also similar; and RP is to TP as Pt is to rP; therefore the rectangle under RP, rP is equal to the rectangle TPt, which is equal to the square of CD, Prop. 52.

PROP.

PROP. LV.

If a right line be drawn from the center of an ellipse or hyperbola perpendicular to any tangent; the rectangle under the perpendicular and the normal, which is terminated by either of the axes, is equal to the square of half the other axis.

DRAW *CY* perpendicular to the tangent; and the rectangle under *CY* and *Pr* will be equal to the square of *CA*; and the rectangle under *CY* and *PR* equal to the square of *CB*. Fig. 55, 56.

From the point *P* draw *PN*, *Pn* perpendicular to the axes; and because the angle at *r* is common to the two right angled triangles *rnP*, *rPt*, the triangle *rnP* is similar to the triangle *rPt*, which is similar to *TCt*, or to *TYC*; therefore *CY* is to *CT* as *Pn*, or *CN*, is to *Pr*; and the rectangle under *CY*, *Pr* is equal to the rectangle under *CN* and *CT*, which is equal to the square of *CA*, Prop. 49. Secondly, because the angle *PRN* is equal to the angle *rPn*, which is equal to the angle *rtP*, the right angled triangles *RPN*, *CtY* are similar; and *CY* is to *Ct* as *PN*, or *Cn*, is to *PR*; therefore the rectangle under *CY*, *PR* is equal to the rectangle under *Ct* and *Cn*, which is equal to the square of *CB*. Prop. 49, and 50.

Cor. 1. The normals are to each other inversely as the squares of the axes by which they are terminated: for, the perpendicular *CY* being common to the two rectangles, *Pr* is to *PR* as the square of *CA* to the square of *CB*.

COR. 2. The perpendicular CY varies inversely as the normal, which is terminated by either of the axes; the rectangle under CY, PR and CY, Pr being each of them equal to a given square.

PROP. LVI.

If a right line be drawn from the center of an ellipse or an hyperbola perpendicular to any tangent; the rectangle under the perpendicular and the semidiameter which is parallel to the tangent is equal to the rectangle under the semi-axes.

FIG. 55, 56. THE same construction remaining as in the preceding propositions, the square of CY is to the rectangle under CY, PR as CY is to PR; and the rectangle under CY, Pr is to the rectangle under PR, Pr, as CY to PR; therefore the square of CY is to the rectangle under CY, PR, or the square of CB, as the rectangle under CY, Pr is to the rectangle under PR, Pr or as the square of CA to the square of CD; therefore CY is to CB as CA to CD; and the rectangle under CY, CD is equal to the rectangle under CA, CB.

FIG. 59, 60. COR. 1. If a parallelogram be formed by drawing tangents through the vertices of any two conjugate diameters; it will be equal to the rectangle under the axes. Let PG, DK be any two conjugate diameters; and draw the tangents through the vertices, which will be parallel to the conjugate diameters; then the four parallelograms DP, PK, KG, GD will be equal to each other; therefore the parallelogram DP is a fourth part of the parallogram VW; and

and the parallelogram DP is equal to the rectangle under PV, CY, or CD, CY, which is equal to the rectangle under AC, CB, a fourth part of the rectangle under the axes.

Cor. 2. Hence the parallelograms, which are formed by drawing tangents through the vertices of conjugate diameters, are equal.

Cor. 3. If DP, PK, KG, GD be joined, the figure $DPKG$ will be a parallelogram, which is half of the parallelogram VW; therefore all the parallelograms, which are formed by joining the vertices of conjugate diameters, are equal.

PROP. LVII.

If two right lines be drawn from any point in an ellipse or an hyperbola to the foci; they will contain a rectangle equal to the square of the semidiameter parallel to the tangent drawn through that point.

FROM any point P in the ellipse or hyperbola draw PS, PH to the foci, and let DK be the diameter which is parallel to the the tangent PY. The rectangle under SP, PH will be equal to the square of CD. Draw the lines SY, HZ from the two foci perpendicular to the tangent; and draw PF perpendicular to the tangent, meeting the diameter DK in F; and PF will be equal to the line which is drawn from the center perpendicular to the tangent. Then, because the rectangle under AC, CB, or PE, CB, is equal to the rectangle under CD, PF, by the preceding proposition, PE is to PF as Fig. 53, 54.

 CD

CD is to *CB*, and becauſe the triangles *SPY*, *PEF*, *HPZ* are ſimilar,
SP is to *SY* as *PE* to *PF*, or as *CD* to *CB*, and
PH is to *HZ* as *PE* to *PF*, or as *CD* to *CB*; therefore the rectangle *SP*, *PH* is to the rectangle *SY*, *HZ*, or the ſquare of *CB*, as the ſquare of *CD* is to the ſquare of *CB*; therefore the rectangle under *SP*, *PH* is equal to the ſquare of *CD*.

PROP. LVIII.

FIG. 51, 53, 54. If a right line *RL* be drawn from the point *R*, where the normal meets the axis of the parabola or the tranſverſe axis of the ellipſe and hyperbola, perpendicular to the diſtance *SP* from the focus to the point of contact: it will cut off from *SP* the ſegment *PL* equal to half the latus rectum.

FIG. 51. FIRST, if the ſection be a parabola, draw *PN* perpendicular to the axis; and becauſe *SR* is equal to *SP*, Cor. 2. Prop. 45. the angle *SPR* is equal to the angle *SRP*; and becauſe the angles *PLR*, *PNR* are right angles, and *PR* is common to the two triangles *PLR*, *PNR*, the triangles are equal, and *PL* is equal to *RN*, or to half the latus rectum, Cor. 4. Prop. 45.

FIG. 53, 54. Secondly, if the ſection be an ellipſe or an hyperbola, let *PR* meet the diameter which is parallel to the tangent in the point *F*. Then the triangles *PRL*, *PEF* will be ſimilar; for the angles at *L* and *F* are right angles, the angle at *P*, Fig. 53. is common, and the angles at *P*, Fig. 54. are vertical; therefore *PE*, or *AC*, is to *PF* as *PR* is to *PL*, and the rect-

rectangle under AC, PL is equal to the rectangle under PF, PR, or to the ſquare of BC, Prop. 55, and AC is to BC as BC is to PL; therefore PL is equal to half the latus rectum, Prop. 7.

Cor. Hence, if PL be taken in any of the conic ſections equal to half the latus rectum, and two right lines be drawn from the points P and L, one of which is perpendicular to the tangent at the point P, and the other perpendicular to PS, they will meet each other in the axis.

PROP. LIX. PROB. VIII.

The diſtance of any point in a conic ſection from the focus, the latus rectum, and the poſition of the tangent at that point being given; to deſcribe the conic ſection.

IF the given diſtance be perpendicular to the tangent, it will be in the direction of the axis, and the conic ſection may be deſcribed by Cor. 1. Prop. 4. But if the diſtance be not perpendicular to the tangent, let SP be the given diſtance, and PY the tangent at the point P. Take PL equal to half the latus rectum; from the points P and L draw the lines PR, LR perpendicular to PS and PY; and the point R where they meet each other is in the axis, by the Cor. to the laſt propoſition. Join SR, which produce indefinitely. Make the angle ZPH equal to the angle SPY; and SR will meet PH in the direction SR, in the oppoſite direction, or it will be parallel to it: in the firſt caſe the conic ſection will be an ellipſe, in the ſecond an hyperbola, and in the third a parabola. If the ſection be an ellipſe or an hyperbola, biſect SH in the point

Fig. 51, 53, 54.

Fig. 53, 54.

point *C*, and draw *CE* parallel to the tangent, meeting the line *PS* in *E*, and *PE* will be equal to half the transverse axis; therefore take *CA*, *CM* each of them equal to *PE*; and *AM* will be the transverse axis. If the section be a parabola, take *SA* equal to a fourth part of the latus rectum, and the point *A* will be the vertex of the axis. The position of the axis, and the distance of the focus from the vertex being found, the conic section may be described by Cor. 1. Prop. 4. or, the transverse axis and the two foci being found in the ellipse and hyperbola, the conic section may be described by Prop. 15, and 16.

FIG. 51.

PROP. LX.

If two ellipses or hyperbolas have a common diameter, and an ordinate be drawn to each of the curves through the same point in the common diameter; the ordinates will be to each other as the conjugate diameters.

FIG. 61, 62.

LET *AP*, *AQ* be two ellipses, or two hyperbolas, having a common diameter *AM*. Through any point *N* in that diameter draw *NP*, *NQ* ordinates to the two curves; and let *CB*, *CD* be the semidiameters which are parallel to *NP*, *NQ*. Then *NP* will be to *NQ* as *CB* is to *CD*: for the square of *NP* is to the square of *CB* as the rectangle *ANM* is to the square of *CA*, or as the square of *NQ* is to the square of *CD*; therefore *NP* is to *CB* as *NQ* is to *CD*; and alternately, *NP* is to *NQ* as *CB* is to *CD*, or as 2*CB* to 2*CD*.

COR.

IG. LI.

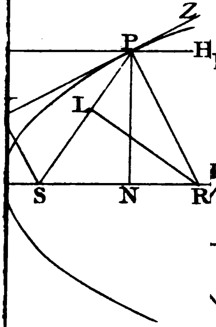


FIG. LVII.

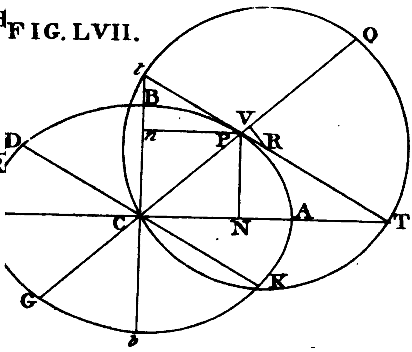


FIG. LIV.

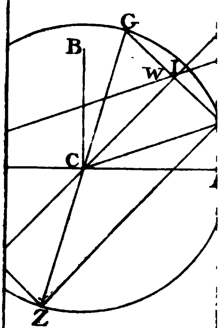
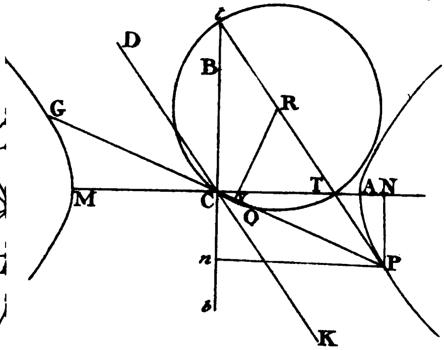
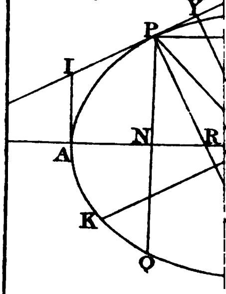
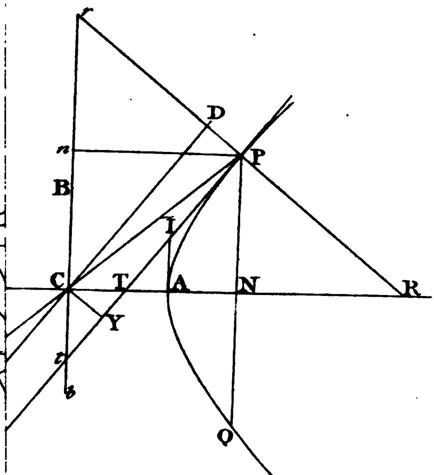


FIG. LVIII.



G. LV.





Cor. If a circle be deſcribed about any diameter of an ellipſe, or an hyperbola, and through any point N in that diameter NR be drawn an ordinate to the circle in the firſt caſe, and a tangent to it in the ſecond; the ſquare of NR will be equal to the rectangle ANM; and if NP be an ordinate to the ellipſe or hyperbola drawn through the ſame point, the ſquare of NR will be to the ſquare of NP as the ſquare of AC is to the ſquare of BC, and NR will be to NP as AC is to BC. Fig. 63, 62.

PROP. LXI.

If two ellipſes or hyperbolas have a common diameter, and an ordinate be drawn to each of the curves through the ſame point in that diameter; the tangents at the extremities of theſe ordinates will meet each other in the diameter.

Let AP, AQ be two ellipſes, or two hyperbolas, having a common diameter AM. Through any point N draw the ſemi-ordinates NP, NQ; and let CB, CD be the ſemidiameters parallel to theſe ordinates. Through the point P draw the tangent PT, meeting the diameter in T, and join TQ, which will be a tangent to the other curve in the point Q: becauſe the tangent PT meets the diameter in T, CN is to CA as CA is to CT, Prop. 49. and therefore TQ touches the other curve, by the ſame propoſition. Fig. 61, 62.

Cor. 1. If a circle and an ellipſe have a common diameter, and a common abſciſſa; the tangents at the extremities of the ordinates will meet each other Fig. 63.

other in the diameter: for a circle may be consider-ed as an ellipse whose axes are equal. Or it may be deduced from a property of the circle. Let *NR* be an ordinate of the circle, and *NP* an ordinate of the ellipse drawn through the same point. Draw *RT* a tangent to the circle, meeting the diameter in *T*, and join *TP* and *CR*. Then because the trian-gles *CRT*, *CNR* are similar, *CN* is to *CR* as *CR* is to *CT*, or *CN* is to *CA* as *CA* is to *CT*; therefore *TP* touches the ellipse in the point *P*.

Cor. 2. If the line *PNQ* drawn through any point *N* in the diameter of a circle, making any given an-gle with *AM*, be bisected in *N*, and *PN* be to *RN*, the ordinate of the circle drawn through the same point, in any given ratio; the points *P*, *Q* will be in an ellipse, of which *AM* is a diameter and *PQ* an ordinate: for draw *BCb* through the center parallel to *PQ*; take *CB* and *Cb* to *CA* in the given ratio; and describe an ellipse, of which *AM*, *Bb* are conjugate diameters. Then because *NP*, or *NQ*, is to *NR* as *CB* to *CA*, and *PQ* is parallel to *Bb*, *PNQ* is an ordinate of that ellipse, and the points *P*, *Q* are in the curve.

PROP.

PROP. LXII.

Let ABM be an ellipſe, of which AM is the transverſe, and Bb the conjugate axis; from any point F in the conjugate axis let a right line FG, which is equal to the ſum or difference of the ſemiaxes CA, CB, be ſo placed as to meet the tranſverſe axis in G; and in FG, produced beyond G when FG is the difference of the ſemiaxes, let GP be taken equal to CB; the point P will be in the ellipſe. Fig. 65.

FROM the point P draw PN perdendicular to the axis AM; and through the center C draw CQ parallel to FP, meeting NP produced in Q. Then CQ is equal to FP, which is equal to CA by the conſtruction; therefore the point Q is in the circumference of a circle, of which C is the center and CA radius; and becauſe the triangles PNG, QNC are ſimilar, PN is to QN as PG is to CQ, or as CB to CA; therefore PN is the ſemi-ordinate, and P is in the ellipſe by the Cor. to the preceding propoſition.

Cor. Hence, if two right lines AM, Bb, of which AM is the greater, biſect each other at right angles in the point C, and a line FP be taken equal to CA, in which the part PG is taken equal to CB, and whilſt the line FP makes one revolution, the point F is always in the line Bb, and G in AM; the point P will deſcribe an ellipſe, of which AM, Bb are the axes.

PROP. LXIII.

If from the vertices of any two conjugate diameters of an ellipſe two ordinates be drawn to the axis; the ſquare of the ſegments of the axis, between the ordinates and the center, are together equal to the ſquare of the ſemi-axis; and the ſquares of the ſemi-ordinates are together equal to the ſquare of the conjugate ſemi-axis.

Fig. 63. LET PG, DK be two conjugate diameters of the ellipſe; and from the vertices P, D draw the ſemi-ordinates PN, DL to the axis AM; the ſquares of CN, CL are together equal to the ſquare of CA; and the ſquares of PN, DL are equal to the ſquare of CB. From the center C, at the diſtance CA, deſcribe a circle, and let the ordinates NP, LD meet the circumference in R and F; and join CR, CF. Through the points R, P draw the tangents RT, PT meeting the axis in T; and becauſe CD is parallel to TP, the triangles CLD, TNP are ſimilar, and

TN is to NP as CL is to LD, and
NP is to NR as LD is to LF; therefore

TN is to NR as CL is to LF; and conſequently the triangles TNR, CLF are ſimilar, and the angle LCF is equal to NTR, which is equal to CRN; and CF being equal to CR, the triangles CLF, CNR are equal, and CL is equal to NR; therefore the ſum of the ſquares of CN, CL is equal to the ſquare of CR, or CA. Secondly, the ſquare of RN is to the ſquare of PN, and the ſquare of FL to the

the ſquare of DL as the ſquare of CA is to the ſquare of CB; therefore the ſum of the ſquares of RN, FL is to the ſum of the ſquares of PN, DL as the ſquare of CA to the ſquare of CB; but the ſum of the ſquares of RN, FL, or of CL, FL is equal to the ſquare of CA; therefore the ſquares of PN, DL are together equal to the ſquare of CB.

PROP. LXIV.

The ſum of the ſquares of any two conjugate diameters of an ellipſe is equal to the ſum of the ſquares of the axes: and the difference of the ſquares of any two conjugate diameters of an hyperbola is equal to the difference of the ſquares of the axes.

FIRST, let ABM be an ellipſe, of which PG, DK are any two conjugate diameters; then, the ſame conſtruction remaining, the ſquare of CP is equal to the ſum of the ſquares of CN, NP, and the ſquare of CD is equal to the ſum of the ſquares of CL, LD; therefore the ſquares of CP, CD are together equal to the ſquares of CN, CL and PN, DL, that is, to the ſquares of CA, CB; and the ſquares of PG, DK are together equal to the ſquares of AM, Bb. FIG. 63.

Secondly, let PAQ be an hyperbola, of which PG, DK are any two conjugate diameters, PG being the greater of the two; and let AM, Bb be the axes. Join PD cutting the aſymptote in l, and draw Pm, Dn perpendicular to the aſymptote. Then, Pl being equal to Dl, Prop. 37. the angles at n and m being right angles, and the angles at l vertical, the FIG. 64.

the triangles Pml, Dnl are equal, and ln is equal to lm: and becauſe the ſquare of CD is leſs than the ſquares of Cl, lD, or Cl, lP, by twice the rectangle Cln, or Clm, and the ſquare of CP is greater than the ſquares of Cl, lP, by twice the rectangle Clm, the difference of the ſquares of CP, CD will be equal to four times the rectangle Clm. But lm is to lP in a conſtant ratio of the coſine of the given angle mlP to radius; and the rectangle Clm is to the conſtant rectangle ClP in the ſame ratio; therefore the difference of the ſquares of CP, CD is invariable, and conſequently equal to the difference of the ſquares of CA, CB; and the difference of the ſquares of PG, DK is equal to the difference of the ſquares of AM, Bb.

PROP. LXV.

The tranſverſe axis is the greateſt of all the diameters of an ellipſe; and the axes of an hyperbola are the leaſt of all its diameters.

FIG. 63. LET PG be any diameter of the ellipſe ABM; through its vertex P draw PNQ an ordinate to the tranſverſe axis AM, and let it meet the circumference of the circle AEM in R. Becauſe AC is greater than BC; RN is greater than PN, and the ſquare of RN greater than the ſquare of PN; therefore the ſum of the ſquares of RN, CN, or the ſquare of CR, is greater than the ſum of the ſquares of PN, CN, or the ſquare of CP; and CR, or CA, is greater than CP; therefore AM is greater than
FIG. 64. PG. Secondly, let PG be any diameter of the hyperbola PAQ, and draw PNQ an ordinate to the axis MA. Then, CNP being a right angled triangle,

gle, CP is greater than CN, which is greater than CA; therefore PG is greater than AM. In the ſame manner it may be proved that, if DK be any diameter of the conjugate hyperbola, CD will be greater than CB, and DK greater than Bb.

COR. Thoſe diameters of the hyperbola which are nearer to the axis are leſs than thoſe which are more remote: for as PN decreaſes, CN decreaſes, Cor. 5. Prop. 6. and therefore CP decreaſes.

PROP. LXVI.

Thoſe diameters which are nearer to the tranſverſe axis of an ellipſe are greater than thoſe which are more remote; the conjugate axis is the leaſt of all the diameters; and any two diameters of the ellipſe or hyperbola, which make equal angles with either of the axes, are equal.

LET ABM be an ellipſe; deſcribe a circle having the axis AM for its diameter; and let the ordinates to the tranſverſe axis AM be produced to meet the circumference of the circle. Becauſe the ſquare of RN is to the ſquare of PN in a conſtant ratio, the difference of the ſquares of RN, PN will be to the ſquare of RN in a conſtant ratio; and if the line RPN be ſuppoſed to move from A to C, RN will increaſe till it becomes equal to EC, or CA, whilſt CN decreaſes; therefore the ſquare of PN, and the difference of the ſquares of RN, PN, which is equal to the difference of the ſquares of CR, CP, will increaſe from A to C; and, CR being FIG. 63.

conſtant,

constant, the square of CP will decrease; and therefore CP will decrease, till it becomes equal to CB, when it will be the least.

Fig. 63, 64. Secondly, let PCG be any diameter of the ellipse or hyperbola; draw PNQ an ordinate to either of the axes, and join CQ. Then, NQ being equal to NP, and CN common to the two right angled triangles CNQ, CNP, the triangles are equal; therefore CQ is equal to CP, and the angle QCN is equal to the angle PCN.

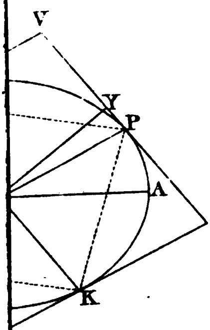
Cor. Hence, if the semidiameters CP, CD be equal, and the angle PCQ be bisected by the line CA, the position of the axis will be determined.

PROP. LXVII.

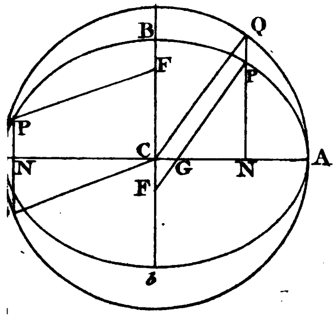
The two diameters of an ellipse, which bisect the right lines joining the vertices of the axes, are equal and conjugate diameters.

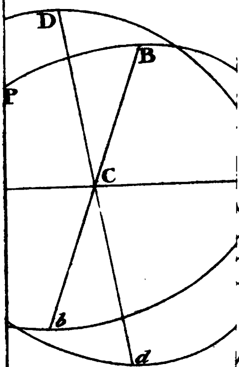
Fig. 66. LET AM, Bb be the axes of an ellipse; join AB, BM, and draw the diameters GP, KD, bisecting the lines AB, BM in the points N and I. Because the lines BA, BM are bisected by the diameters GP, KD, they are ordinates to these diameters, Cor. 1. Prop. 33. and because AM is bisected in C, and AB in N; CN is parallel to MB; therefore GP, KD are conjugate diameters: and the angle ACB being a right angle, it will be in a semicircle, of which AB is the diameter, and N the center; therefore NA will be equal to NC, and the angle NCA equal to the angle NAC, which is equal to the alternate angle ACK; and therefore the diameters PG, KD are equal, by the preceding proposition.

PROP.



IG. LXV.





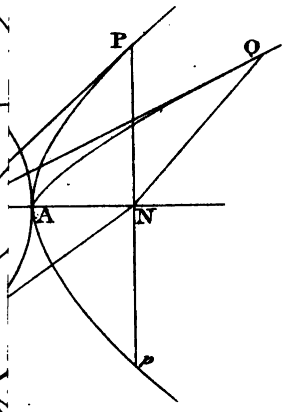
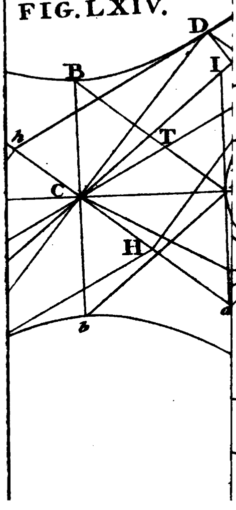
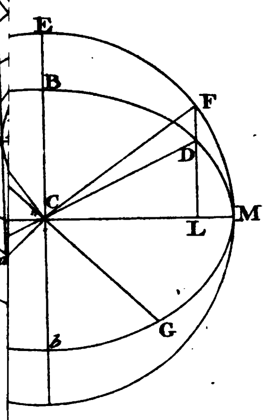


FIG. LXIV.





PROP. LXVIII. PROB. IX.

To find the axes of a given conic section.

FIRST, let the section be an ellipse or an hyperbola, and find any two diameters, by Cor. 4. Prop. 33. cutting each other in the point C: from the center C, at the distance CP, which is the greater semidiameter in the hyperbola, and which is greater than the lesser semidiameter in the ellipse, but less than the greater, describe a circle; which will cut the curve, or the opposite curves, in the points P and G; and because the ellipse and hyperbola have each of them another diameter equal to PG, it will also cut the curve, or opposite curves, in two other points Q and K. Join CQ, and draw CA bisecting the angle PCQ, which will be one of the axes, Cor. Prop. 66. Join PQ, which will be bisected by the line CA in N; it is, therefore, an ordinate to the axis; and if BCb be drawn through C parallel to PQ, meeting the ellipse in the points B, b, it will be the other axis; the length of which, in the hyperbola, may be determined in the following manner; from the center C, at the distance CA, describe a circle; from the point N draw NR a tangent to the circle, and take CB, Cb each of them a fourth proportional to the lines RN, PN and CA; and Bb will be the conjugate axis, Cor. Prop. 60. Fig. 67, 68. Fig. 68.

Secondly, let the section be a parabola, of which find any diameter BC; and if it bisects its ordinates at right angles, it is the axis; if it does not bisect them at right angles, through any point C draw QCP perpendicular to BC, meeting the parabola in the points Q, P; bisect QP in the point N, and draw Fig. 69.

NA

NA parallel to CB, which will be the axis: for NA is a diameter which bisects QP; therefore QP is an ordinate to that diameter; and because it is perpendicular to NA; NA is the axis.

PROP. LXIX.

If two ellipses or two hyperbolas have a common axis, and an ordinate be drawn through the same point in the axis to each of the curves; the areas included between the common abscissa, the ordinates, and the two curves, also the whole areas of the ellipses will be to each other as the conjugate axes.

Fig. 70, 71. LET AP, AQ be two ellipses, or two hyperbolas; take any abscissa AN, which is not greater than half the axis of the ellipse, and draw the ordinates NP, NQ. The areas ANP, ANQ are to each other as the conjugate axes. Let the abscissa AN be divided into any number of equal parts, AE, EF, FG, GN; through the points E, F, G draw the ordinates ERI, FSK, GTL, and complete the parallelograms, AR, AI, ES, EK, &c. also from the points I, K, L draw Ii, Kk, Ll parallel to AN. Then it is evident that the difference between the circumscribed parallelograms AI, EK, FL, GQ and the inscribed parallelograms Ei, Fk, Gl is equal to the parallelogram GQ; and if parallelograms be inscribed, in the same manner, in the other figure APN, the difference between these and the circumscribed parallelograms would be equal to the parallelogram GP; therefore the differences between each series of parallelo-

parallelograms and the areas AQN, APN will be leſs than the parallelograms GQ, GP; and becauſe the parallelogram GP is to the parallelogram GQ as NP is to NQ, and each parallelogram in the figure APN is to the correſponding parallelogram in the figure AQN in the ſame ratio; the ſum of all the parallograms in the figure APN is to the ſum of all in the figure AQN as NP is to NQ; and the area APN will be to the area AQN as the ſum of the parallelograms in APN to the ſum in AQN: for if not, let the parallelograms APN be to the parallelograms AQN as the area APN to ſome ſpace X greater or leſs than the area AQN. Firſt, let the ſpace X be greater; then if the baſes AE, EF, FG, GN be continually biſected, and the parallelograms completed as above, the parallelograms GP, GQ, will be diminiſhed in the ratio of two to one at each biſection, and the difference between each ſeries of parallelograms and the areas APN, AQN will be diminiſhed more than half; therefore the difference between the circumſcribed parallelograms AQN and the area AQN may be made leſs than any given ſpace; let it be leſs than the difference between the area AQN and the ſpace X, and the circumſcribed parallelograms AQN will be leſs than the ſpace X; but the ſum of the parallelograms APN is to the ſum of the parallelograms AQN as the area APN is to the ſpace X, and the ſum of the parallelograms APN is greater than the area APN; therefore the ſum of the parallelograms AQN is greater than X; which is impoſſible. Secondly, let the parallelograms APN be to the parallelograms AQN as the area APN to ſome ſpace X leſs than the area AQN, and let the difference between the inſcribed parallelograms AQN and the area AQN be made leſs than the difference between the area AQN and X; then the parallelo-

grams AQN will be greater than X, but the inscribed parallelograms APN are less than the area APN; therefore the parallelograms AQN are less than X; which is impossible. Therefore the area APN is to the area AQN as the parallelograms APN to the parallelograms AQN, or as NP is to NQ, that is, as the conjugate axes Prop. 60.

Fig. 70. If the sections be two ellipses, and the abscissa MN be greater than half the axis, the area ACB is to the area ACD as CB to CD; and by division, the area $CNPB$, is to the area $CNQD$ as CB to CD; therefore by composition, the area MPN is to the area MQN, and the area MBA to the area MDA as CB is to CD, and consequently the whole areas $MBAb$, $MDAd$ are in the same ratio.

Cor. 1. If a circle be described about the transverse axis of an ellipse; the area of the circle will be to the area of the ellipse, as the transverse axis is to the conjugate axis.

Fig. 70, 71. Cor. 2. The area of an ellipse is equal to that of a circle, whose diameter is a mean proportional between the two axes: for let $ABMb$ be an ellipse, and let $ADMd$ be a circle having the transverse axis for a diameter; let HV be a mean proportional between AM, Bb, and describe the circle having HV for a diameter. Then, because AM, HV, Bb are continual proportionals, the square of AM is to the square of HV as AM is to Bb, that is, as the area of the circle $ADMd$ to the area of the ellipse $ABMb$; and the square of AM is to the square of HV as the circle $ADMd$ to the circle HXV; therefore the area of the circle HXV is equal to the area of the ellipse $ABMb$.

Cor. 3. The areas of any two ellipses are to each other as the rectangles under their axes: for the area of the ellipse $ABMb$ is to the circle HXV as the

the rectangle under AM, Bb is to the square of HV, that is, in a ratio of equality; and alternately, the area of the ellipse $ABMb$ is to the rectangle under AM, Bb as the circle HXV is to the square of HV; but all circles are as the squares of their diameters; therefore the area of the ellipse is to the rectangle under the axes in a given ratio.

PROP. LXX.

If two parabolas have a common axis, and an ordinate be drawn through the same point in the axis to each of the curves; the areas of the parabolas, included between the common abscissa, the ordinates, and the two curves, are to each other in a subduplicate ratio of the latera recta.

LET AP, AQ be two parabolas, having a common axis AN; through any point N in the axis draw the ordinates NP, NQ; and let L and M be the latera recta of the two parabolas AP, AQ. Then the square of NP is to the square of NQ as the rectangle under AN and L to the rectangle under AN and M, or as L is to M; therefore NP is to NQ in the subduplicate ratio of L to M; and the ordinates NP, NQ being to each other in a constant ratio, it may be proved, in the same manner as in the preceding proposition, that the areas APN, AQN are to each other in the same ratio. FIG. 73.

PROP. LXXI.

If any ordinate and abſciſſa of a parabola be completed into a parallelogram; the area of the parabola, included between the ordinate and the curve, is to the parallelogram as 2 to 3.

Fig. 74. LET *AN* be any diameter of the parabola, and *PQ* an ordinate to that diameter; through the point *A* draw *BC* parallel to *PQ*; and through the points *P*, *Q* draw *PB*, *QC* parallel to *NA*. The area of the parabola *PAQ* will be to the parallelogram *PBCQ* as 2 to 3. Join *PA*, *AQ*; and through the points *P*, *Q* draw the tangents *PT*, *QT*, meeting the diameter in *T*: through the points *E*, *G* draw the diameters *ED*, *GK*, which will biſect the lines *PA*, *AQ* in *D*, *K*, Cor. 2. Prop. 34. and through the vertices draw the tangents *RL*, *MV*; join *PF*, *FA*, and *AH*, *HQ*. Then, *NA* being equal to *AT*, and *PQ* equal to twice *EG*, the triangle *PAQ* will be double the triangle *TEG*; for the ſame reaſon the triangles *PFA*, *AHQ* will be double the triangles *ERL*, *GMV*; therefore the inſcribed figure *PFAHQ* will be double the external figure *TRLAMV*, and the ſame proportion holds whatever be the number of triangles inſcribed: but, by proceeding in this manner, the difference between the inſcribed figure *PFAHQ* and the area *PAQ*, and the difference between the external figure and the area *TPAQ* will each of them become leſs than any given ſpace: for the triangle *PAQ* being half of the parallelogram *PBCQ*, it is greater than half of the area *PAQ*; and, for the ſame reaſon, the

tri-

triangles *PFA*, *AHQ* are each of them greater than half of the areas *PFA*, *AHQ*; and *PE* being equal to *ET*, the triangle *PEA* is equal to the triangle *EAT*, and the triangle *TEG* is half of the two triangles *TPA*, *TQA*, and therefore more than half of the external area *TPAQ*; and, for the ſame reaſon, the triangles *ERL*, *GMV* are more than half of the areas *EPFA*, *GAHQ*; therefore, by inſcribing triangles as above, the difference between each rectilinear figure and the parabolic area is diminiſhed more than half at each operation, and therefore may be made leſs than any given ſpace; and the parabolic area *PAQ* will be double the area *TPAQ*: for if it were greater than double that area by any given ſpace *S*, a ſeries of triangles might be inſcribed in the area *PAQ*, which would differ from it by a ſpace leſs than *S*, and this ſeries of triangles would be more than double the area *TPAQ*; and conſequently more than double the correſponding figure inſcribed in the area *TPAQ*; which is imimpoſſible. If the area *PAQ* were leſs than double the area *TPAQ*, this area would be greater than half the area *PAQ* by ſome ſpace *S*; and therefore ſince a ſeries of triangles might be inſcribed in the area *TPAQ*, which would differ from it by a ſpace leſs than *S*, this ſeries would be greater than half of the area *PAQ*, and therefore greater than half of the ſeries of correſponding triangles in the area *PAQ*; which is impoſſible. Therefore it follows that the area *PAQ* is neither more nor leſs than double the area *TPAQ*, and conſequently the area *PAQ* is to the whole triangle *PTQ* as 2 to 3. But the triangle *PTQ* is equal to the parallelogram *PBCQ*; therefore the area *PAQ* is to the parallelogram *PBCQ* as 2 to 3.

COR. Hence, the area *PFAN* being two thirds

be

tote CI; it may be proved, in the same manner, that the areas $AEFQ$, $QFLM$ are equal.

Cor. 2. Hence, if the segments of the asymptote be taken in continued proportion, the areas, beginning from the first line DP, will be in arithmetical proportion.

DEFINITIONS.

XXVIII. A circle is said to touch a conic section in any point, when the circle and the conic section have a common tangent in that point.

XXIX. If a circle touches a conic section in any point, so that no other circle can be drawn between this circle and the conic section, it is said to have the same curvature with the section in the point of contact, and it is called the circle of Curvature.

LEMMA III.

Fig. 77. If two tangents PT, VT be drawn at the extremities of any chord of a circle, and from any point Q in the circumference a chord Qq be drawn parallel to one of the tangents TP, cutting the chord PV in the point N, and from the points Q, q the lines QH, qh be drawn parallel to the other tangent TV, meeting the chord PV in H and h; the square of QN will be equal to the rectangle under PN, VH; and the square of qN will be equal to the rectangle under PN, Vh.

JOIN

I.

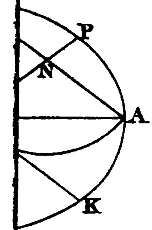
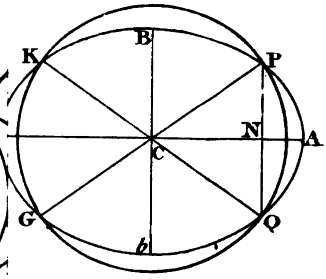


FIG. LXVII.



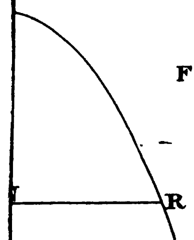
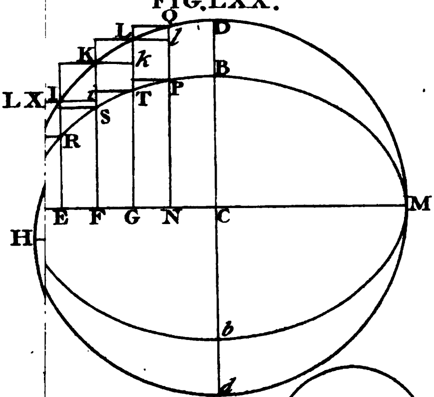
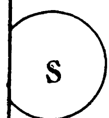
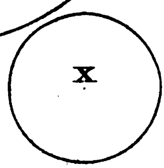


FIG. LX

FIG. LXX.







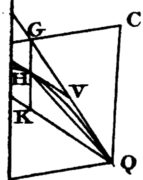


FIG. LXXVI.

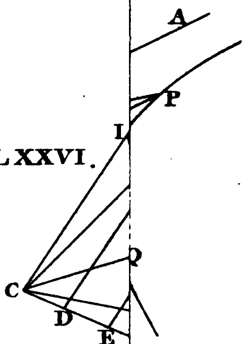
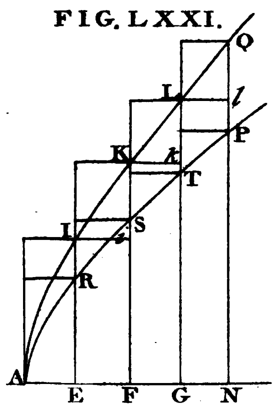


FIG. LXXI.



JOIN QP, QV, alſo qP, qV; and becauſe the lines QN, QH are parallel to TP, TV, the triangles TPV, QNH are ſimilar; therefore QN is equal to QH, the angle QHN is equal to the angle QNH, and the angle QHV equal to the angle QNP; and the angle PQN being equal to the alternate angle QPT, which is equal to the angle QVP in the alternate ſegment, the triangles QPN, QVH are ſimilar; therefore PN is to QN as QH, or QN, is to VH, and the rectangle under PN, VH is equal to the ſquare of QN. In the ſame manner it may be proved, that the triangles PqN, qVh are ſimilar; therefore PN is to qN as qh, or qN, is to Vh, and the rectangle under PN, Vh is equal to the ſquare of qN.

COR. If the line Qq, which is always parallel to TP, be ſuppoſed to move from V to P; the points H, N, h, and the points Q, q will continually approach to P, in which point they will all coincide.

PROP. LXXIV.

If a circle touches a conic ſection in any point, and cuts off from the diameter which paſſes through that point a ſegment greater than its parameter, a part of the circumference on each ſide of the point of contact will be wholly without the conic ſection; and if it cuts off from the diameter a ſegment leſs than its parameter, a part of the circumference on each ſide of the point of contact will be wholly within the conic ſection.

FIG. 78. *Caſe* 1. LET the conic ſection be a parabola, of which PV is any diameter; let PR be a common tangent to the parabola and the circle in the point P; take PR equal to the parameter of that diameter, and through the point R draw RL parallel to PV; let the chord Qq, which is parallel to RP, be produced to meet RL in L, and from the points Q, q draw QH, qh parallel to the tangent TV. Firſt, let the circle cut off a ſegment PV greater than the parameter, and take VB equal to PR, or NL. Then let the line Qq be ſuppoſed to move from V to P, and when the point H comes to B, VH will be equal to VB, or NL; and the rectangle under PN, VH, or the ſquare of NQ, will be equal to the rectangle under PN, NL, which is equal to the ſquare of the ſemi-ordinate of the parabola, Prop. 43. therefore NQ will be equal to the ſemi-ordinate, and Q is a point in the parabola. But

But when the point H is any where between B and P, VH will be greater than NL; therefore NQ, and consequently Nq will be greater than the semi-ordinate, and the arc of the circle QPq will be wholly without the parabola. Secondly, let PV be less than PR, and take Vb equal to PR. When h comes to b, Vh will be equal to NL; therefore Nq will be equal to the semi-ordinate, and q is a point in the parabola; but when h is any where between b and P, Vh will be less than NL; therefore Nq, and consequently NQ will be less than the semi-ordinate, and the arc QPq will be wholly within the parabola.

Case 2. Let the section be an ellipse, of which PG is any diameter, and take PR in the tangent at P equal to the parameter of that diameter; join RG, and let the chord Qq meet RG in the point L. First, let the chord PV be greater than the parameter, and take VB equal to PR; then, NL being less than PR, VH will be equal to NL before H comes to B, when NQ will be equal to the semi-ordinate, and Q will be a point in the ellipse; when H is any where between this point and P, VH will be greater than NL; therefore NQ, and consequently Nq will be greater than the semi-ordinate, and the arc QPq will be wholly without the ellipse. Secondly, if PV be less than the parameter, take Vb equal to PR; from P draw PO parallel to RL, meeting the chord Qq in O; then OL will be equal to PR; and as the point N approaches to P, the point O will approach to N; and as h moves from b towards P, bh increases, and NO decreases; therefore bh will be equal to NO before h comes to P, and Vh will be equal to LN; therefore Nq will be equal to the semi-ordinate; from this point to P, bh will be greater than NO, and Vb less than NL; there- FIG. 79.

fore Nq, and consequently NQ will be less than the semi-ordinate, and the arc QPq will be wholly within the ellipse.

FIG. 80. *Case* 3. Let the section be an hyperbola, of which PG is any diameter, in the tangent at P take PR equal to the parameter; join RG, and produce the chord Qq till it meet GR in L; and NL will be greater than PR. First, let the chord PV be greater than the parameter; take VB equal to PR, and from P draw PO parallel to RL. When the point H is between B and P, but nearer to B, BH will be less than NO; and as H approaches to P, BH increases, and NO decreases; therefore BH will be equal to NO before H comes to P, and VH will be equal to NL; therefore NQ will be equal to the semi-ordinate of the hyperbola; from this point to P, BH being greater than NO, VH will be greater than NL; therefore NQ, and consequently Nq will be greater than the semi-ordinate, and the arc QPq will be wholly without the hyperbola. Secondly, let PV be less than the parameter, and take Vb equal to PR. When h comes to b, Vh will be equal to PR; it will, therefore, be equal to NL before h comes to b, when Nq will be equal to the semi-ordinate of the hyperbola; from this point to P, Vh will be less than NL; therefore Nq, and consequently NQ will be less than the semi-ordinate, and the arc QPq will be wholly within the hyperbola.

COR. 1. If a circle touches a conic section, and cuts off from the diameter which passes through the point of contact a segment equal to its parameter, no other circle can be drawn between this circle and the conic section: for if a greater circle be described, it will cut off from the diameter a segment greater than its parameter, and a part of the cir-

circumference on each side of the point of contact will be wholly without the conic section, it will also be without the former circle; and if a less circle be described, it will cut off from the diameter a segment less than its parameter; it will, therefore, be within the conic section on each side of the point of contact, and it will fall within the former circle.

Cor. 2. The chord, which the circle of curvature cuts off from the diameter of a parabola, is equal to four times the distance of the vertex of that diameter from the focus.

Cor. 3. The chord, which the circle of curvature cuts off from a diameter of an ellipse or an hyperbola, is a third proportional to that diameter and its conjugate.

Cor. 4. If two conic sections have the same parameter, and the ordinates of each make the same angle with the diameter, they will have the same circle of curvature.

PROP.

PROP. LXXV.

The circle of curvature at the vertex of the transverse axis of an ellipse or hyperbola, or at the vertex of the axis of a parabola, falls wholly within the conic section: but the circle of curvature at the vertex of the conjugate axis of an ellipse falls wholly without the ellipse.

Fig. 78, 79, 80. THE same construction remaining, the chord *PV* at the vertex of an axis will be perpendicular to the tangent; it will, therefore, be a diameter of the circle; and *Qq* being perpendicular to *PV*, it will be bisected in *N*, and *QH*, *qh* will coincide with *QN*, *qN*; and the squares of *QN*, *qN* will each of them be equal to the rectangle *PNV*: and if the section be a parabola or an hyperbola, Fig. 78, 80. *VN* will be less than *VP*, or *PR*, and consequently less than *NL*; therefore *NQ* will be less than the semi-ordinate, and the circle will fall wholly within the section. If the section be an ellipse, Fig. 79. *PN* is to *NO* as *PG* is to *PR*; therefore *PN* will be greater or less than *NO*, and *VN* less or greater than *NL*, according as *PG* is greater or less than *PR*. At the vertex of the transverse axis *PG* is greater than *PR*; therefore *VN* is less than *NL*, and *NQ* is less than the semi-ordinate; but at the vertex of the conjugate axis *PG* is less than *PR*; therefore *VN* is greater than *NL*, and *NQ* is greater than the semi-ordinate. Therefore in the former case the circle falls wholly within, and in the latter, without the ellipse.

PROP.

PROP. LXXVI.

The circle of curvature at the vertex of any diameter of a conic ſection, which is not an axis, cuts the ſection in that point; it alſo cuts it in another point, which may be determined.

Caſe 1. LET the ſection be a parabola; and, the ſame conſtruction remaining, draw *PM* parallel to the tangent *TV*, meeting the circle in the point *M*. Then, *MP* being parallel to *qh*, if *q* be any where in the arc *PM*, between *P* and *M*, *Vh* will be greater than *VP*, or *NL*; therefore *Nq* will be greater than the ſemi-ordinate. When *q* is at *M*, *Vh* will be equal to *VP*, or *NL*, and *M* is a point in the parabola. If *q* be any where between *M* and *V*, *Vh* will be leſs than *NL*, and *Nq* leſs than the ſemi-ordinate; and if *Q* be any where between *P* and *V*, *VH* will be leſs than *NL*, and *NQ* leſs than the ſemi-ordinate. Therefore the circle cuts the parabola in the points *P* and *M*; the arc *PqM* is without, and the arc *PQM* is within the parabola. FIG. 78.

Caſe 2. If the ſection be an ellipſe; draw *PW* parallel to the tangent *TV*. Then it is evident, that, if *q* be any where in the arc *PW*, *Vh* will be greater than *PR*, and conſequently greater than *NL*; therefore *Nq* is greater than the ſemi-ordinate, and the arc *PqW* is without the ellipſe. In the tangent *TV* take *TE* to *TV* as the diameter *PG* to the parameter *PR*, and draw *PE* cutting the circle in *M*; then the arc *PQM* will be within the ellipſe, and the circle will cut the ellipſe in *M*: for take FIG. 79.

any

any point Q between P and M; join PQ, and produce it till it meet TV in F, and let HQ meet the tangent TP in I. Becauſe IH is parallel to TV, IH is to TV as PI is to PT, or as IQ to TF; and alternately, IH is to IQ as TV is to TF; but, NQ being parallel to PI,

PH is to PN as IH to IQ, as TV to TF, and PN is to NO as GP to PR, as TE to TV; therefore PH is to NO as TE is to TF. Hence, whilſt TF is leſs than TE, NO will be leſs than PH, and conſequently NL greater than VH, and the ſemi-ordinate greater than NQ. If Q be at M, NO will be equal to PH, and NL equal to VH; therefore M is a point in the ellipſe. But if TF be greater than TE, NL will be leſs than VH, and the ſemi-ordinate leſs than NQ; therefore the circle cuts the ellipſe in M.

If F be on the other ſide of V, or q be any where between V and W, it may be ſhown in the ſame manner, that Vh will be greater than NL, and conſequently Nq greater than the ſemi-ordinate. Therefore the circle cuts the ellipſe in the points P and M; the arc PWM is without, and the arc PQM is within the ellipſe.

FIG. 80. *Caſe* 3. If the ſection be an hyperbola, VH being leſs than VP, or PR, and conſequently leſs than NL, NQ is leſs than the ſemi-ordinate. In VT produced take TE to TV as PG is to PR; join EP, and produce it to meet the circle in M. Take any point q between P and M; join qP, and produce it to meet TE in F; and let the tangent TP meet qh in I. Then, qh being parallel to VF, hI is to TV as PI is to PT, as Iq to TF; and alternately, hI is to Iq as TV is to TF; but

hP is to PN as hI to Iq, as TV to TF, and PN is to NO as PG to PR, as TE to TV; therefore

hP

hP is to NO as TE is to TF. Hence, whilſt TF is leſs than TE, NO will be leſs than hP, and conſequently NL leſs than Vh; therefore Nq is greater than the ſemi-ordinate. If q be at M, TF will be equal to TE, and Vh will be equal to NL; therefore M is a point in the hyperbola. Draw PW parallel to TV; then if q be any where between M and W, TF will be greater than TE, and NL greater than Vh; therefore Nq is leſs than the ſemi-ordinate; but when q comes to W, Vh is equal to VP, or PR, which is leſs than NL, and from W to V, Vh will be leſs than VP. Therefore the circle cuts the hyperbola in the points P and M; the arc PqM is without, and the arc PQM is within the hyperbola.

PROP. LXXVII.

The chord of the circle of curvature, which is drawn from the point of contact through the focus of a parabola, is equal to that which is cut off from the diameter; and half the radius of the circle is a third proportional to the perpendicular drawn from the focus upon the tangent, and the diſtance of the point of contact from the focus.

LET PV be the chord which is cut off from the diameter; draw PSW through the focus, meeting the circle in W; and draw the diameter PR; join VW, RW; biſect PR in O; and draw SY from the focus perpendicular to the tangent. Then, FIG. 81.

the angle SPY being equal to the angle VPZ, Prop. 25. the angles in the alternate ſegments will be equal, that is, the angle PVW equal to the angle PWV, and PW is equal to PV. Secondly, the triangles RPW, SPY being ſimilar, RP is to PW, or $4SP$, as SP is to SY; therefore half PO is to SP as SP is to SY.

Cor. 1. Hence the radius of curvature is equal to $\frac{2SP^2}{SY}$.

Cor. 2. Becauſe the radius of curvature varies as the ſquare of SP directly, and as SY inverſely, and SP varies as the ſquare of SY, Cor. 1. Prop. 45. the radius of curvature will vary as the cube of the perpendicular SY, or as the cube of the normal, Cor. 3. Prop. 45.

PROP. LXXVIII.

The radius of the circle of curvature at the vertex of any diameter of an ellipſe, or an hyperbola, is a third proportional to the perpendicular drawn from the center upon the tangent, and the conjugate ſemidiameter; and the chord which is drawn from the point of contact through the focus is a third proportional to the tranſverſe axis, and the conjugate diameter.

Fig. 82, 83. LET PV be the chord which is cut off from the diameter; draw the diameter of the circle PR, and from the center O draw OT perpendicular to PV, which will biſect PV in T; draw the conju-

conjugate diameter DCK, cutting PR in F, and PF will be equal to the perpendicular drawn from the center G upon the tangent; draw the chord PW through the focus S, and let it meet the conjugate diameter in E; and join RW. Because the triangles PFC, PTO are similar,

PF is to PC as PT is to PO, and
PC is to CD as CD is to PT; therefore
PF is to CD as CD is to PO.

Secondly, because the triangles PEF, PRW are similar,

PW is to PR as PF to PE, or as $2PF$ to $2PE$, and
PR is to $2CD$ as $2CD$ is to $2PF$; therefore
PW is to $2CD$ as $2CD$ is to $2PE$, or $2AC$.

Cor. 1. Hence the radius of curvature is equal to $\frac{CD^2}{PF}$; and the chord which is drawn through the focus is equal to $\frac{2CD^2}{AC}$.

Cor. 2. Because CD is to AC as CB to PF, Prop. 56. the square of CD will vary inversely as the square of PF; therefore the radius of curvature will vary inversely as the cube of PF, or directly as the cube of the normal which is terminated by either of the axes, Cor. 2. Prop. 55.

PROP. LXXIX.

FIG. 81, 82, 83. If the right line PY touches a conic ſection in any point P, and PS be drawn from the point of contact to the focus, and SY from the focus perpendicular to the tangent; the radius of the circle of curvature at the point P will be to half the latus rectum in the triplicate ratio of SP to SY.

FIRST, let the conic ſection be a parabola; and becauſe a fourth part of the latus rectum is a

FIG. 81. third proportional to SP, SY, Prop. 45. and half the radius of curvature is a third proportional to SP, SY, Prop. 77. if L be the latus rectum,

$2SP$ is to half L in the duplicate ratio of SP to SY, and

PO is to $2SP$ as SP is to SY; therefore

PO is to half L in the triplicate ratio of SP to SY.

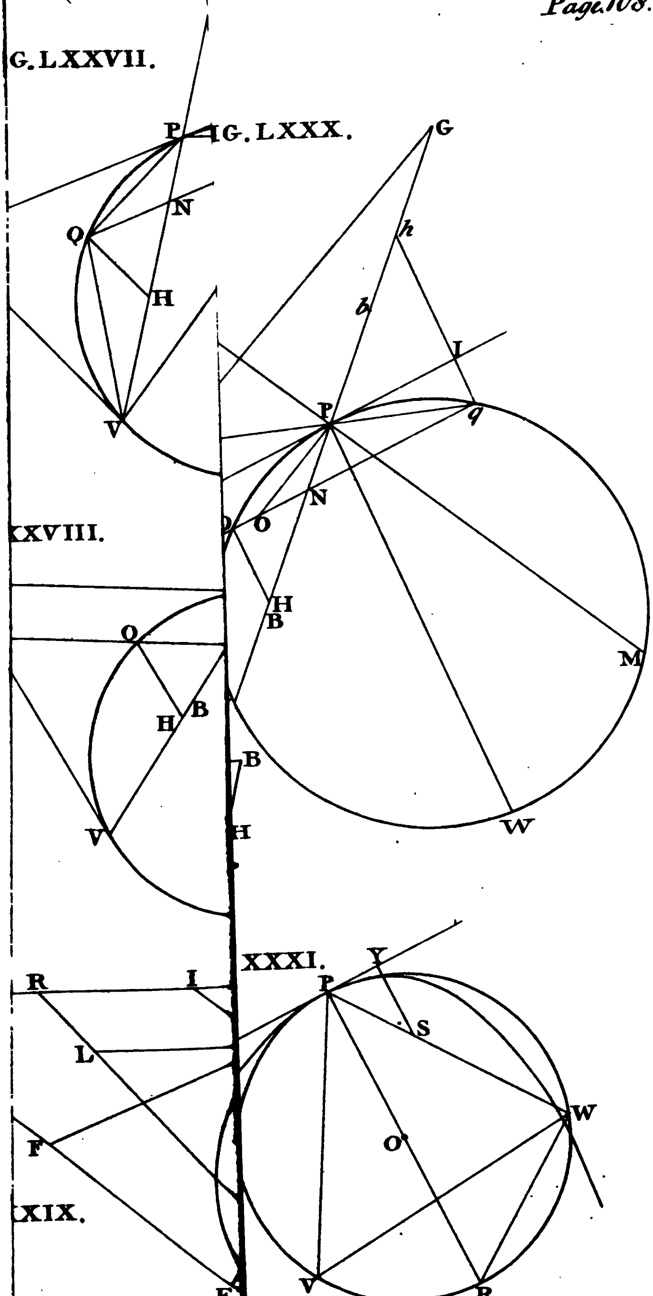
FIG. 82, 83. Secondly, if the conic ſection be an ellipſe or an hyperbola, the latus rectum is a third proportional to the tranſverſe and conjugate axes, and the chord of curvature, which is drawn through the focus, is a third proportional to the tranſverſe axis, and the conjugate diameter, Prop. 78. therefore

half PW is to AC as CD^2 to AC^2, and

AC is to half L as AC^2 to CB^2; therefore

$\frac{1}{2}PW$ is to half L as CD^2 to CB^2, that is, Prop. 57. in the duplicate ratio of SP to SY; and becauſe the triangles RPW, SPY are ſimilar, PO is to half PW as SP to SY; therefore PO is to half L in the triplicate ratio of SP to SY.

COR.


G. LXXVII.
IG. LXXX.
XXVIII.
XXXI.
XIX.

Cor. Hence the radius of the circle of curvature in all the conic ſections is equal to $\frac{\frac{1}{2}L \times SP^3}{SY^3}$.

DEFINITIONS.

XXX. If an indefinite right line paſſing through any fixed point A, without the plane of the circle CGB, be carried round the whole circumference of the circle, each of the ſurfaces generated by this motion is called a Conical Surface. Fig. 84.

XXXI. The ſolid contained by the conical ſurface and the circle CGB is called a Cone.

XXXII. The point A is called the Vertex of the Cone.

XXXIII. The circle CGB, the Baſe of the Cone.

XXXIV. Any right line drawn from the vertex to the circumference of the baſe, is called a Side of the Cone.

XXXV. The right line AD paſſing through the vertex and the center of the baſe, which is produced indefinitely, is called the Axis.

XXXVI. A right Cone is that whoſe axis is perpendicular to the baſe.

XXXVII. A ſcalene Cone is that whoſe axis is inclined to the baſe.

PROP.

PROP. LXXX.

If a cone be cut by a plane paſſing through its vertex, the ſection will be a triangle.

FIG. 84. LET *ABGC* be a cone, of which *AD* is the axis; and let *GB* be the common ſection of the baſe of the cone and the cutting plane; join *AB*, *AG*. When the generating line comes to the two points *B* and *G*, it is evident that it will coincide with the right lines *AB*, *AG*; they are, therefore, in the ſurface of the cone, and they are in the plane which paſſes through the points *A*, *B* and *G*; therefore the triangle *ABG* is the common ſection of the cone, and the plane which paſſes through its vertex.

PROP. LXXXI.

If a cone be cut by a plane parallel to its baſe, the ſection will be a circle, the center of which is in the axis.

FIG. 84. LET *HFK* be the ſection made by a plane parallel to the baſe of the cone, and let *ACB*, *ADG* be two ſections of the cone, made by any two planes paſſing through the axis *AD*; let *KH*, *EF* be the common ſections of the plane *HFK* and the triangles *ACB*, *ADG*. Then, becauſe the planes *HFK*, *BGC* are parallel, *EH*, *EF* will be parallel to *DB*, *DG*, and *EH* will be to *DB* as *AE* is to *AD*, or as *EF* is to *DG*; and alternately, *EH* is to *EF* as *DB* to *DG*; but *DB* is equal to *DG*;

DG; therefore EH is equal EF, and, for the ſame reaſon, EF is equal to EK; therefore HFK is a circle, of which E is the center.

PROP. LXXXII.

If a ſcalene cone $ABDC$ be cut through the axis by a plane perpendicular to the baſe, making the triangle ABC, and from any point L in the right line AC, LM be drawn in the plane of the triangle, ſo that the angle ALM may be equal to the angle ABC, and the cone be cut by another plane paſſing through LM, perpendicular to the triangle ABC; the common ſection $LPMQ$ of this plane and the cone will be a circle. Fig. 86.

TAKE any point N in the right line LM; through N draw FNG parallel to CB; and let $FPGQ$ be a ſection parallel to the baſe, paſſing through FG; then the two planes $FPGQ$, $LPMQ$ being perpendicular to the plane ABC, their common ſection PNQ is perpendicular to FNG; therefore PN is equal to NQ, and the ſquare of PN equal to the rectangle FNG; but, the angle ALM being equal to the angle ABC, or AGF, and the angles at N being vertical, the triangles FLN, MGN are ſimilar, and MN is to NG as NF to NL; therefore the rectangle MNL is equal to the rectangle FNG, or to the ſquare of PN. Therefore the ſection $LPMQ$ is a circle, of which LM is a diameter.

This ſection is called a Subcontrary ſection.

PROP.

PROP. LXXXIII.

If a cone be cut by a plane, which does not pass through the vertex, and which is neither parallel to the base, nor to the plane of a subcontrary section; the common section of the plane and the surface of the cone will be an ellipse, a parabola, or an hyperbola, according as a plane passing through the vertex parallel to the cutting plane falls without the cone, touches it, or falls within the cone.

Fig. 86, 85, 87. LET *ABDC* be any cone; and let *STV* be the common section of a plane passing through its vertex and the plane of the base, which will fall without the base, will touch it, or it will fall within; let *PMQ* be a section made by a plane parallel to *ASV*; through the center *O* of the base draw *OT* perpendicular to *SV*, meeting the circumference of the base in the points *B* and *C*; let a plane pass through the points *A*, *B* and *C*, meeting the plane *ASV* in the line *AT*, the surface of the cone in *AB*, *AC*, and the plane of the section *PMQ* in *LM*; then *LM* will be parallel to *TA*, the planes *SAV*, *PMQ* being parallel; it will meet the side *AB* in *M*, and it will meet the other side *AC*, Fig. 86. in *L*, within the cone, it will be parallel to it in Fig. 85. and it will meet it Fig. 87. produced beyond the vertex in *K*. Take any point *N* in the line *LM*; let *FPGQ* be a plane passing through *N* parallel to the base; and let *FNG*, *PNQ* be the common sections of this plane and the planes *ABC*, *PMQ*:

PMQ: then *PNQ* will be parallel to *SV*, and *GF* parallel to *BT*; and *BT* being perpendicular to *SV*, *FNG* is perpendicular to *PNQ*; therefore *PN* is equal to *NQ*, and the ſquare of *PN* is equal to the rectangle *FNG*. Firſt, if the line *STV* be without the baſe, through the points *M* and *L* draw *MH*, *LK* parallel to *CB*; then, becauſe the triangles *LNF*, *LMH* are ſimilar, as alſo the triangles *MNG*, *MLK*, Fig. 86.

LN is to *FN* as *LM* is to *HM*, and
NM is to *NG* as *LM* is to *LK*; therefore

the rectangle *LNM* is to the rectangle *FNG*, or the ſquare of *PN*, as the ſquare of *LM* is to the rectangle under *HM*, *LK*; which ratio is the ſame, wherever the point *N* be taken; therefore the ſection *LPMQ* is an ellipſe, of which *LM* is a diameter, and *PNQ* an ordinate, Cor. 2. Prop. 53.

Secondly, if the line *STV* touches the circumference of the baſe in *C*; let *DLE* be the common ſection of the baſe and the plane *PMQ*, which is parallel to *PN*, and perpendicular to *BLC*; and the rectangle *BLC* is equal to the ſquare of *DL*; therefore the ſquare of *PN* is to the ſquare of *DL* as the rectangle *FNG* to the rectangle *BLC*, or, becauſe *NG* is equal to *LC*, as *FN* to *BL*; but, the triangles *MNF*, *MLB* being ſimilar, *FN* is to *BL* as *MN* to *ML*; therefore the ſquare of *PN* is to the ſquare of *DL* as *MN* to *ML*; and the ſection *DME* is a parabola, of which *ML* is a diameter, and *PNQ* an ordinate, Cor. 2. Prop. 44. Fig. 85.

Laſtly, let the line *STV* fall within the baſe; through the vertex *A* draw *AH* parallel to *GF*; and becauſe the triangles *MNF*, *MHA* are ſimilar, as alſo the triangles *KNG*, *KHA*, Fig. 87.

MN is to *NF* as *MH* is to *HA*, and
KN is to *NG* as *KH* is to *HA*; therefore

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the rectangle *MNK* is to the rectangle *FNG*, or the square of *PN*, as the rectangle under *MH*, *KH* is to the square of *HA*, that is, in a constant ratio; therefore the section *DME* is an hyperbola, of which *MK* is a diameter, and *PNQ* an ordinate, Cor. 2. Prop. 30.

LXXXIII.

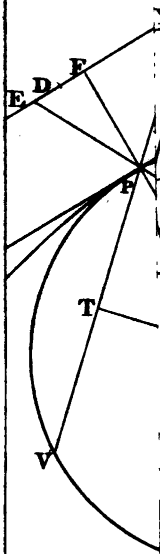
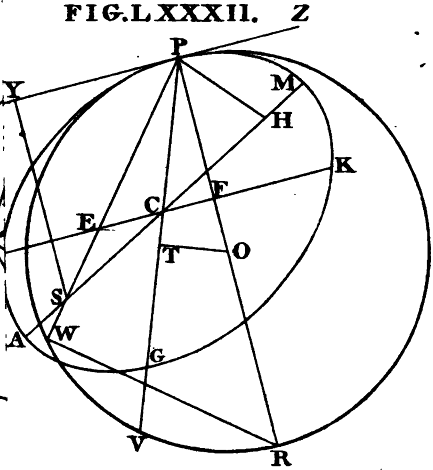


FIG. LXXXII.



XXXVI.

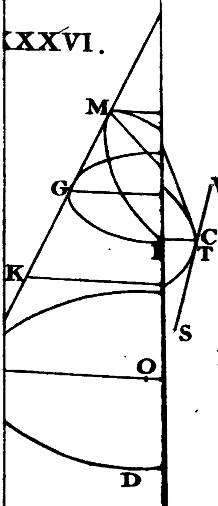
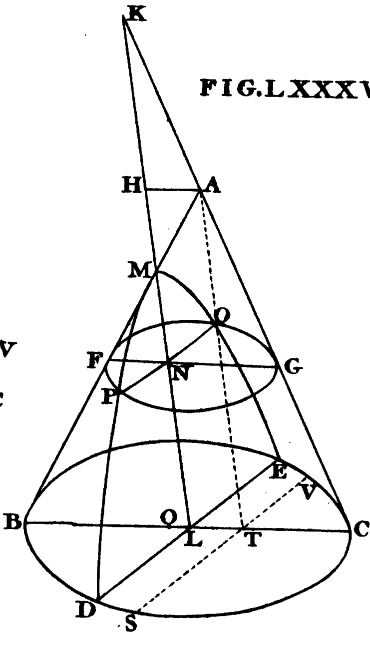


FIG. LXXXVII.



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ERRATA.

PAGE 5. line 3. and 8. *for NL, read NQ.*
— line 16. *for SP, read SM.*
6. line 8. *for* Conjugale, *read* Conjugate.
7. line 8. *for* PROB. *read* PROB. I.
9. line 10. *for* ſrom, *read* from.
18. line 5. from the bottom, *for SD, read SN.*
31. line 19. *for* ſemi-conjugate axis, *read* conjugate ſemi-axis.
41. line 5. *for* parallel, *read* parallel to.
— line 17. *for* ſquare, *read* ſquare of.
42. line 17. *for* $a^2 R^2$, *read* $a^2 S^2$, and *for* $a^2 S^2$, *read* $a^2 R^2$.
46. line 4. *for* equal, *read* equal to.
47. line 8. from the bottom, *dele* to.
53. line 11. *for the laſt PH, read VH.*
— line 11. from the bottom, *for VDH, read VDh.*
86. line 11. *for CD, read CQ.*
89. line 6. *for* parallograms, *read* parallelograms.
93. catch word, *for* be, *read* of.
104. line 10. from the bottom, *for* tnan, *read* than.
111. line 1. *for* equal, *read* equal to.